On mechanical systems with a Lie group as configuration space

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À la mémoire du Professeur Jean Leray, en témoignage d'admiration et de respect

1. Introduction

Several physically important mechanical systems have a configuration space which may be identified with a Lie group. Let us give some examples.

Example 1. Let us consider a massive point particle moving in the physical space E, of dimension 3. The configuration space is the affine space E. Once a particular point O of E has been chosen as origin (that point being looked at as a reference configuration of the system) we can consider E as a vector space, therefore as a Lie group with the addition of vectors as composition law.

More generally, let us consider a system of n massive point particles moving in the physical 3-dimensional space E. The configuration space is now $E \times \cdots \times E$, product of n copies of E, one for each particle. We allow two distinct particles to occupy the same position in space at a given time (otherwise we should take as configuration space an open subset of $E \times \cdots \times E$). As above, once a particular point $O = (O_1, \ldots, O_n)$ of $E \times \cdots \times E$ is chosen, the configuration space of the system can be considered as a vector space, therefore as a Lie group.

Example 2. Let us consider a massive rigid body moving in the Euclidean 3-dimensional space E. We assume that the body has at least three distinct points not on the same straight line. Let S_0 be a particular position of the body in E, which will be considered as a reference configuration of the system. Then for any other configuration, *i.e.*, for any other position S of the rigid body in E, there exists a unique element g of the group $\mathcal{E}(E)$ of Euclidean displacements of E which maps S_0 onto S. Therefore, once a reference configuration is chosen, the configuration space of the system can be identified with the Lie group $\mathcal{E}(E)$.

Example 3. Let us particularize slightly Example 2, by assuming that our rigid body has a point which remains fixed in space, and that it rotates around that point. We take that point as origin. As above, once a reference configuration is chosen, the configuration space of the system can be identified with the subgroup of elements $g \in \mathcal{E}(E)$ such that $g(0) = 0$, *i.e*, with the linear group $SO(E, 0)$ (the space E, with 0 as origin, being now considered as an Euclidean vector space).

Example 4. Consider an ideal, incompressible fluid which fills a vessel V fixed in the physical space E, and flows in that vessel. We assume that V is a compact, connected part of E, bounded by a smooth surface ∂V. To describe the configuration space of the system, we introduce the set of all fluid particles; it is an abstract set $\mathcal{V}$, with a smooth three-dimensional manifold structure with boundary, and a volume 3-form. A configuration of the system is a diffeomorphism $\varphi : \mathcal{V} \to V$ such that the pullback of the natural (that means determined by the Euclidean structure) volume 3-form v of V is equal to the 3-form given of $\mathcal{V}$. Let $\varphi_0 : \mathcal{V} \to V$ be a reference configuration of the system. For any other configuration $\varphi : \mathcal{V} \to V$, there exists a unique element g of the group $Diff(V, v)$ of volume-preserving diffeomorphisms of V such that $\varphi = g \circ \varphi_0$. Conversely, for any $g \in Diff(V, v)$, $g \circ \varphi_0$ is a configuration of the system. Therefore the configuration space of the system can be identified with $Diff(V, v)$, which is an infinite-dimensional Lie group.

Example 5. As we shall see later, the well known Korteweg-de Vries equation on the circle S^1 can be considered as the Euler equation of a mechanical system whose configuration space is an infinite-dimensional Lie group, the Virasoro-Bott group.

Among these systems, the most remarkable are those whose phase space is a non-Abelian Lie group (examples 2 to 5). These systems share remarkable properties which derive from the fact that a Lie group G acts on itself by two distinct actions: the left and right translations. These two actions can be canonically lifted into two actions,

$$\widehat{L} : G \times T^*G \to T^*G \quad \text{and} \quad \widehat{R} : T^*G \times G \to T^*G$$

of the Lie group G on its cotangent bundle T^*G. These two actions are Hamiltonian with respect to the canonical symplectic structure of the cotangent bundle T^*G, and have Ad^*-equivariant momentum maps $J_L : T^*G \to \mathcal{G}^*$ and $J_R : T^*G \to \mathcal{G}^*$. Moreover, the Hamiltonian $H : T^*G \to \mathbb{R}$ of the system is invariant, in some cases under the action $\widehat{L}$, in other cases under the other action $\widehat{R}$, and still in other cases under the restriction of $\widehat{L}$ (or of $\widehat{R}$) to a subgroup of G. As a consequence, the Hamiltonian vector field X_H on T^*G can be projected,

(i) by J_R into $\mathcal{G}^*$ when H is $\widehat{L}$-invariant,
(ii) by J_L into $\mathcal{G}^*$ when H is $\widehat{R}$-invariant,
(iii) by the momentum map of an action of a new Lie group G_1, which is a semi-direct product of G with an Abelian group, when H is invariant under the restriction of $\widehat{L}$ (or of $\widehat{R}$) to a suitable subgroup of G.

That projected vector field defines a differential equation (on $\mathcal{G}^*$ in cases (i) and (ii), and on $\mathcal{G}_1^*$ in case (iii)), called the Euler equation. That equation is Hamiltonian with respect to the Lie-Poisson structure of $\mathcal{G}^*$ (or of $\mathcal{G}_1^*$).

These remarkable properties were discovered by Euler [3] around 1765 for Examples 3 and 4 (the motion of a rigid body around a fixed point, and the motion of an ideal, incompressible fluid). They were expressed with the modern concepts of Lie groups and Lie algebras, and generalized, by V. Arnol'd [1] in 1966. The construction of the group G_1 and of its action on T^*G, needed when the Hamiltonian H is invariant by the restriction of $\widehat{L}$ (or of $\widehat{R}$) to a subgroup of G only, appears in the works of Iacob and Sternberg [5], Marsden, Ratiu and Weinstein [7], Guillemin and Sternberg [4]. The recent book by Arnol'd and Khesin [2] fully develops applications of Lie groups to Hydrodynamics.

In Section 2 we discuss the actions $\widehat{L}$ and $\widehat{R}$ of a Lie group G on its cotangent bundle T^*G, and we obtain the expressions and properties of their momentum maps J_L and J_R.

In Section 3 we present a slight generalization, useful for mechanical systems involving a magnetic field. The phase space of such systems is the cotangent bundle to their configuration manifold equipped with a symplectic form which differs from the canonical symplectic form of a cotangent bundle: it is the sum of the canonical symplectic form and of the pull-back of a closed 2-form on the configuration space.

In Section 4 we discuss the construction of the Lie group G_1 and its action on T^*G, when one of the two actions $\widehat{L}$ or $\widehat{R}$ is restricted to a subgroup of G.

Finally in Sections 5 to 7 we develop the examples briefly sketched above.

2. The right and left actions of a Lie group on its cotangent bundle

Let G be a Lie group. We will denote by $\mathcal{G}$ its Lie algebra, and by $\mathcal{G}^*$ the dual of $\mathcal{G}$ For any $g \in G$, we denote by $L_g : G \to G$ the left translation $L_g(h) = gh$, and by $R_g : G \to G$ the right translation $R_g(h) = hg$. We denote by $TL_g : TG \to TG$ and $TR_g : TG \to TG$ the canonical lifts to the tangent bundle TG of L_g and R_g, respectively. We define the maps $\widehat{L}_g : T^*G \to T^*G$ and $\widehat{R}_g : T^*G \to T^*G$, as the transpose of $TL_{g^{-1}}$ and $TR_{g^{-1}}$, respectively:

$$\widehat{L}_g = {}^t(TL_{g^{-1}})\,, \qquad \widehat{R}_g = {}^t(TR_{g^{-1}})\,.$$

We recall that the Liouville 1-form α on the cotangent bundle T^*G is given by the formula, where $w \in T(T^*G)$,

$$\bigl\langle \alpha\bigl(p_{T^*G}(w)\bigr), w\bigr\rangle = \bigl\langle p_{T^*G}(w), Tq_G(w)\bigr\rangle\,,$$

where $p_{T^*G} : T(T^*G) \to T^*G$ and $q_G : T^*G \to G$ are the canonical projections.

We denote by $\Omega = d\alpha$ the canonical symplectic form on T^*G.

Theorem 1. *The maps*

$$\widehat{L} : G \times T^*G \to T^*G\,, \quad \widehat{L}(g,\xi) = \widehat{L}_g\xi\,, \quad \text{and} \quad \widehat{R} : T^*G \times G \to T^*G\,, \quad \widehat{R}(\xi,g) = \widehat{R}_g\xi\,,$$

are two commuting Hamiltonian actions of the Lie group G on the symplectic manifold (T^*G,Ω), respectively on the left and on the right. They admit as Ad**-equivariant momentum maps, respectively, the maps $J_L : T^*G \to \mathcal{G}^*$ and $J_R : T^*G \to \mathcal{G}^*$,*

$$J_L(\xi) = \widehat{R}_{(q_G(\xi))^{-1}}\xi\,, \quad J_R(\xi) = \widehat{L}_{(q_G(\xi))^{-1}}\xi\,.$$

Proof. The actions L and R of G on itself by left and right translations satisfy, for all g and $h \in G$,

$$L_g \circ L_h = L_{gh}\,, \qquad R_g \circ R_h = R_{hg}\,, \quad L_g \circ R_h = R_h \circ L_g\,.$$

Therefore L is an action of the left and R an action on the right, and these two actions commute. Since $\widehat{L}$ and $\widehat{R}$ are the canonical lifts to the cotangent bundle T^*G of the actions L and G, respectively, they satisfy, for all g and $h \in G$,

$$\widehat{L}_g \circ \widehat{L}_h = \widehat{L}_{gh}\,, \qquad \widehat{R}_g \circ \widehat{R}_h = \widehat{R}_{hg}\,, \quad \widehat{L}_g \circ \widehat{R}_h = \widehat{R}_h \circ \widehat{L}_g\,.$$

This means that $\widehat{L}$ is an action on the left, $\widehat{R}$ an action on the right, and that these two actions commute.

For any $g \in G$, $\widehat{L}_g$ is the canonical lift to the cotangent bundle T^*G of the diffeomorphism $L_g : G \to G$; therefore, it satisfies

$$(\widehat{L}_g)^*\alpha = \alpha\,, \quad (\widehat{L}_g)^*\Omega = \Omega\,.$$

This shows that $\widehat{L}$ is a symplectic action of G on (T^*G, Ω). For the same reason, $\widehat{R}$ is a symplectic action of G on (T^*G, Ω).

For every $X \in \mathcal{G}$, we denote by $X^L_{T^*G}$ and $X^R_{T^*G}$ the vector fields on T^*G defined by

$$X^L_{T^*G}(\xi) = \frac{d}{dt}\left(\widehat{L}_{\exp(tX)}\xi\right)\Big|_{t=0}\,, \quad X^R_{T^*G}(\xi) = \frac{d}{dt}\left(\widehat{R}_{\exp(tX)}\xi\right)\Big|_{t=0}\,.$$

They are called the fundamental vector fields on T^*G associated to the element X of the Lie algebra $\mathcal{G}$, for the actions $\widehat{L}$ and $\widehat{R}$, respectively. In order to prove that the action L is Hamiltonian and admits as a momentum map the map $J_L : T^*G \to \mathcal{G}^*$, we must check that for any $X \in \mathcal{G}$, the fundamental vector field $X^L_{T^*G}$ is Hamiltonian and admits as Hamiltonian the function $X \circ J_L$ (we consider here X as a linear form on the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$). In other words, we must check that

$$i(X^L_{T^*G})\Omega = -d(X \circ J_L)\,.$$

Since the action $\widehat{L}$ leaves invariant the Liouville 1-form α, we have

$$\mathcal{L}(X^L_{T^*G})\alpha = i(X^L_{T^*G})d\alpha + di(X^L_{T^*G})\alpha = 0\,,$$

therefore, since $\Omega = d\alpha$,

$$i(X^L_{T^*G})\Omega = -di(X^L_{T^*G})\alpha\,.$$

According to the definition of the Liouville 1-form α, we have, for any $\xi \in T^*G$,

$$i(X^L_{T^*G})\alpha(\xi) = \left\langle \xi, Tq_G(X^L_{T^*G}(\xi))\right\rangle\,.$$

But since $\widehat{L}$ is the canonical lift to T^*G of the action of G on itself by left translations, we have

$$Tq_G(X^L_{T^*G}(\xi)) = \frac{d}{dt}\left(L_{\exp(tX)}q_G(\xi)\right)\big|_{t=0} = TR_{q_G(\xi)}X\,.$$

Therefore, since $J_L(\xi) = \widehat{R}_{q_G(\xi)^{-1}}(\xi)$,

$$\langle \xi, Tq_G(X^L_{T^*G}(\xi))\rangle = \langle \widehat{R}_{q_G(\xi)^{-1}}\xi, X\rangle = X\circ J_L(\xi)\,.$$

So we obtain

$$i(X^L_{T^*G})\Omega = -d(X\circ J_L)\,.$$

A similar calculation shows that

$$i(X^R_{T^*G})\Omega = -d(X\circ J_R)\,.$$

We have proved that the actions $\widehat{L}$ and $\widehat{R}$ are Hamiltonian and admit as momentum maps the maps J_L and J_R, respectively.

Finally its very expression shows that J_L is equivariant for the $\widehat{L}$-action of G on T^*G and the Ad^*-action of G on $\mathcal{G}^*$. Similarly, J_R is equivariant for the $\widehat{R}$-action of G on T^*G and the Ad^*-action of G on $\mathcal{G}^*$. The action of G on T^*G being tacitely assumed to be $\widehat{L}$ when we consider J_L and $\widehat{R}$ when we consider J_R, we will say, in short, that both J_L and J_R are Ad^*-equivariant. ▯

Remarks. The momentum maps J_L and J_R are submersions. For any $\xi \in T^*G$, we observe that $J_L(\xi)$ is the unique point where the orbit of ξ under the action $\widehat{R}$ meets $\mathcal{G}^*$, considered as the cotangent space to G at the unit element. Similarly $J_R(\xi)$ is the unique point where the orbit of ξ under the action $\widehat{L}$ meets $\mathcal{G}^*$. Therefore the orbits of the $\widehat{L}$-action are the level sets of J_R, and the orbits of the $\widehat{R}$-action are the level sets of J_R.

Theorem 2. *Let n be the dimension of the Lie group G. For each point $\xi \in T^*G$, the tangent spaces at ξ to the $\widehat{L}$-orbit and to the $\widehat{R}$-orbit of that point are two n-dimensional vector subspaces of $T_\xi(T^*G)$, symplectically orthogonal to each other. The momentum map $J_L : T^*G \to \mathcal{G}^*$ is a Poisson map when $\mathcal{G}^*$ is equipped with the Lie-Poisson bracket, called the minus Lie-Poisson bracket, given by the formula (where f and g are two smooth functions on $\mathcal{G}^*$ and η a point in $\mathcal{G}^*$),*

$$\{f,g\}_-(\eta) = -\langle \eta, [df(\eta), dg(\eta)]\rangle\,.$$

Similarly, the momentum map $J_R : T^*G \to \mathcal{G}^*$ is a Poisson map when $\mathcal{G}^*$ is equipped with the Lie-Poisson bracket, called the plus Poisson bracket, opposite of the previous one,

$$\{f,g\}_+(\eta) = \langle \eta, [df(\eta), dg(\eta)]\rangle\,.$$

Application to a G-invariant Hamiltonian system on T^*G.

Let $H : T^*G \to \mathbb{R}$ be a smooth function, and X_H the associated Hamiltonian vector field on T^*G, such that

$$i(X_H)\Omega = -dH\,.$$

Let us assume that H is invariant under the action $\widehat{L}$ of G on T^*G. Of course, similar results would hold, *mutatis mutandis*, if H were invariant under the action $\widehat{R}$. Since the $\widehat{L}$-orbits are the level sets of J_R, there exists a unique smooth function $\widehat{H} : \mathcal{G}^* \to \mathbb{R}$ such that

$$H = \widehat{H} \circ J_R \,.$$

Since J_R is a Poisson map (when $\mathcal{G}^*$ is equipped with the plus Lie-Poisson bracket), the Hamiltonian vector field X_H on T^*G projects, under the map J_R, onto $\mathcal{G}^*$, and its projection is the Hamiltonian vector field $X_{\widehat{H}}$ associated with the Hamiltonian $\widehat{H}$ on the Poisson manifold $(\mathcal{G}^*, \{\ ,\ \}_+)$. The differential equation on $\mathcal{G}^*$,

$$\frac{d\varphi(t)}{dt} = X_{\widehat{H}}\bigl(\varphi(t)\bigr) \,,$$

defined by the vector field $X_{\widehat{H}}$, is called the *Euler equation*.

Remark. Under the same assumption as above, Noether's theorem shows that the momentum map J_L is constant on each integral curve of X_H. In order to make easier the determination of these integral curves, one may use the well known Marsden-Weinstein reduction procedure [8]. Let η be an element of $\mathcal{G}^*$. To determine the integral curves of X_H on which the (constant) value taken by J_L is η, we first consider $J_L^{-1}(\eta)$, which contains all these integral curves. Since J_L is a submersion, $J_L^{-1}(\eta)$ is a submanifold of T^*G. Let G_η be the stabilizer of η (for the coadjoint action of G on $\mathcal{G}^*$). Since J_L is Ad^*-equivariant, the restriction of the $\widehat{L}$-action to the subgroup G_η leaves $J_L^{-1}(\eta)$ invariant. The quotient manifold $J_L^{-1}(\eta)/G_\eta$, that means the set of orbits of the $\widehat{L}$-action of G_η on $J_L^{-1}(\eta)$, has a reduced symplectic structure. Moreover, there exists a unique smooth function $\widetilde{H} : J_L^{-1}(\eta)/G_\eta \to \mathbb{R}$ such that

$$H\,\big|_{J_L^{-1}(\eta)} = \widetilde{H} \circ \pi \,,$$

where $\pi : J_L^{-1}(\eta) \to J_L^{-1}(\eta)/G_\eta$ is the canonical projection. That projection maps the integral curves of X_H contained in $J_L^{-1}(\eta)$ onto the integral curves of the Hamiltonian vector field $X_{\widetilde{H}}$, on the reduced symplectic manifold $J_L^{-1}(\eta)/G_\eta$.

So we see that two different procedures can be used in order to make easier the determination of the integral curves of X_H:

— the use of the Poisson map $J_R : T^*G \to \mathcal{G}$, which maps these curves onto the integral curves of the Euler equation on the Poisson manifold $\mathcal{G}^*$,

— the use of a level set $J_L^{-1}(\eta)$ of the momentum map J_L and of the canonical projection $\pi : J_L^{-1}(\eta) \to J_L^{-1}(\eta)/G_\eta$, which maps the integral curves of X_H contained in $J_L^{-1}(\eta)$ onto the integral curves of the Hamiltonian vector field $X_{\widetilde{H}}$, on the reduced symplectic manifold $J_L^{-1}(\eta)/G_\eta$.

These two procedures are essentially equivalent: the connected components of the reduced symplectic manifolds of Marsden and Weinstein are symplectomorphic to the symplectic leaves of the Poisson manifold $(\mathcal{G}^*, \{\ ,\ \}_+)$, which are exactly the connected components of the coadjoint orbits. However, the use of the Euler equation has the advantage of giving at once, by a single operation, the same result as that given by looking at all the reduced symplectic manifolds for all the values of η.

3. Taking into account a magnetic field

For mechanical systems made of material bodies moving in an electromagnetic field, one has to replace the canonical symplectic form Ω of the cotangent bundle T^*G by another symplectic form, sum of the canonical form Ω and of the pull-back of a closed 2-form on the configuration space G (see, for example, the book by J.-M. Souriau [9]). This leads us to the following generalization.

As in Section 2, G is a Lie group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ the dual space of $\mathcal{G}$. Let $\theta : G \to \mathcal{G}$ be a smooth map such that, for all g and $h \in G$,

$$\theta(gh) = \mathrm{Ad}^*_g\, \theta(h) + \theta(g)\,, \tag{$*$}$$

and such that its differential at the unit element, $\Theta = T_e\theta$, is skew-symmetric: for all X and $Y \in \mathcal{G}$ (identified with T_eG),

$$\bigl\langle T_e\theta(X), Y\bigr\rangle = -\bigl\langle T_e\theta(Y), X\bigr\rangle\,.$$

We set

$$\Theta(X, Y) = \bigl\langle T_e\theta(X), Y\bigr\rangle\,.$$

We can consider Θ as a bilinear skew-symmetric 2-form on the Lie algebra $\mathcal{G}$, or as a left invariant differential 2-form on the Lie group G. As a consequence of $(*)$, Θ is closed:

$$d\Theta = 0\,.$$

We now consider the 2-form on T^*G:

$$\Omega_\theta = d\alpha + q_G^*\Theta\,,$$

where α is the Liouville 1-form on T^*G. That 2-form is closed and nondegenerate, *i.e.*, symplectic. Since Θ is left invariant, Ω_θ is invariant under the $\widehat{L}$ action defined in Section 2: for every $g \in G$, we have

$$\widehat{L}^*_g\Omega_\theta = \Omega_\theta\,.$$

However, Ω_θ is not invariant under the other action, $\widehat{R}$, defined in Section 2. We define a new map $\widehat{R}^\theta : T^*G \times G \to T^*G$ by setting, for all $g \in G$, $\xi \in T^*G$,

$$\widehat{R}^\theta(\xi, g) = \widehat{R}^\theta_g(\xi) = \widehat{R}_g\xi + \widehat{L}_{q_G(\xi)g}\theta(g^{-1})\,.$$

An easy calculation shows that $\widehat{R}^\theta$ is an action of G on T^*G on the right which commutes with the action $\widehat{L}$. In other words it satisfies, for all g and $h \in G$,

$$\widehat{R}^\theta_g \circ \widehat{R}^\theta_h = \widehat{R}^\theta_{hg}\,, \qquad \widehat{L}_g \circ \widehat{R}^\theta_h = \widehat{R}^\theta_h \circ \widehat{L}_g\,.$$

Moreover, the action $\widehat{R}^\theta$ leaves invariant the symplectic 2-form Ω_θ: for all $g \in G$,

$$(\widehat{R}^\theta_g)^*\Omega_\theta = \Omega_\theta \,.$$

The following theorem generalizes Theorem 1 of Section 2.

Theorem 3. *The maps*

$$\widehat{L} : G \times T^*G \to T^*G \,, \quad \widehat{L}(g,\xi) = \widehat{L}_g\xi \,, \quad \text{and} \quad \widehat{R}^\theta : T^*G \times G \to T^*G \,, \quad \widehat{R}(\xi, g) = \widehat{R}_g\xi \,,$$

are two commuting Hamiltonian actions of the Lie group G on the symplectic manifold (T^*G, Ω_θ), respectively on the left and on the right. They admit as momentum map, respectively, the maps $J^\theta_L : T^*G \to \mathcal{G}^*$ and $J_R : T^*G \to \mathcal{G}^*$,

$$J^\theta_L(\xi) = \widehat{R}_{q_G(\xi)^{-1}}\xi + \theta\bigl(q_G(\xi)\bigr) \,, \quad J_R(\xi) = \widehat{L}_{q_G(\xi)^{-1}}\xi \,.$$

The momentum map J^θ_L is equivariant with respect to the action $\widehat{L}$ of G on T^*G and the affine action of G on $\mathcal{G}^*$:

$$(g, \eta) \mapsto \mathrm{Ad}^*_g \eta + \theta(g) \,,$$

while the momentum map J_R is equivariant with respect to the action $\widehat{R}^\theta$ of G on T^*G on the right and the affine action of G on $\mathcal{G}^*$ on the right:

$$(\eta, g) \mapsto \mathrm{Ad}^*_{g^{-1}} \eta + \theta(g^{-1}) \,.$$

Remarks. Let us set, for all $g \in G$, $\eta \in \mathcal{G}^*$,

$$a_\theta(g, \eta) = \mathrm{Ad}^*_g \eta + \theta(g) \,.$$

Then $a_\theta : G \times \mathcal{G}^* \to \mathcal{G}^*$ is the action of G on $\mathcal{G}^*$ on the left for which J^θ_L is equivariant. The action of G on $\mathcal{G}^*$ on the right for which J_R is equivariant is the same action a_θ, but with g replaced by g^{-1}, in order to have an action on the right; in other words, it is the action on the right

$$(\eta, g) \mapsto a_\theta(g^{-1}, \eta) \,.$$

In [6] the two actions of G on $\mathcal{G}^*$ for which J^θ_L and J_R are equivariant were not so simply related, because at that time we made different sign conventions, leading to the replacement of J^θ_L by its opposite. The conventions made here now seem to us more natural.

As in Section 2, the orbits of $\widehat{L}$ are the level sets of J_R, and the orbits of $\widehat{R}^\theta$ are the level sets of J^θ_L. As a consequence, we have the following theorem, which generalizes Theorem 2 of Section 2.

Theorem 4. *Let n be the dimension of the Lie group G. For each point $\xi \in T^*G$, the tangent spaces at ξ to the $\widehat{L}$-orbit and to the $\widehat{R}^\theta$-orbit of that point are two n-dimensional* *vector subspaces of $T_\xi(T^*G)$, symplectically orthogonal to each other (with respect to the symplectic form $\Omega_\theta(\xi)$). The momentum map $J_L^\theta : T^*G \to \mathcal{G}^*$ is a Poisson map when T^*G is equipped with the symplectic 2-form Ω_θ and $\mathcal{G}^*$ with the modified Lie-Poisson bracket (callet the θ-modified minus Lie-Poisson bracket) given by the formula (where f and g are two smooth functions on $\mathcal{G}^*$ and η a point in $\mathcal{G}^*$),*

$$\{f,g\}_-^\theta(\eta) = -\bigl\langle \eta, [df(\eta), dg(\eta)]\bigr\rangle + \Theta\bigl(df(\eta), dg(\eta)\bigr)\,.$$

Similarly, the momentum map $J_R : T^*G \to \mathcal{G}^*$ is a Poisson map when T^*G is equipped with the symplectic 2-form Ω_θ and $\mathcal{G}^*$ with the modified Lie-Poisson bracket, opposite of the previous one (called the θ-modified plus Lie-Poisson bracket),

$$\{f,g\}_+^\theta(\eta) = \bigl\langle \eta, [df(\eta), dg(\eta)]\bigr\rangle - \Theta\bigl(df(\eta), dg(\eta)\bigr)\,.$$

4. Reduction of one of the actions $\widehat{L}$ or $\widehat{R}$ to a subgroup

Several mechanical systems with a Lie group G as configuration space have a Hamiltonian $H : T^*G \to G$ which is invariant under the restriction of the action $\widehat{L}$ to a subgroup of G, rather than under the full action $\widehat{L}$ of G on T^*G. For example, the motion of a heavy rigid body with a fixed point (example 3 of Section 1) has the Lie group $SO(E,0)$ as configuration space; when the body is submitted to the gravitational force, its Hamiltonian is no more invariant under the full action $\widehat{L}$, but only under the restriction of that action to the one-dimensional subgroup of $SO(E,0)$ made by the rotations around the vertical axis through the fixed point of the body.

A very remarkable property allows us to obtain, even in that case, an Euler equation (at least when the subgroup of G for which the restricted $\widehat{L}$-action leaves invariant the Hamiltonian is the stabilizer of a point for a linear representation of G in a finite dimensional vector space). Let us describe more fully that property.

As in the preceding sections, G is a Lie group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ the dual space of $\mathcal{G}$. The cotangent bundle T^*G will be equipped with its canonical symplectic 2-form $\Omega = d\alpha$, as in Section 2. (Of course, it is possible to extend the results when T^*G is equipped with a modified symplectic 2-form Ω_θ, as in Section 3.)

Let $\rho : G \to GL(E)$ be a linear representation of G in a finite-dimensional vector space E. We denote by $\rho^* : G \to GL(E^*)$ the contragredient representation of G in the dual space E^* of E. We recall that for each $x \in E$, $\zeta \in E^*$ and $g \in G$, we have

$$\langle \rho_g^* \zeta, x\rangle = \langle \zeta, \rho_{g^{-1}} x\rangle\,.$$

Let η be a point in E^*, and let

$$G_\eta = \{\, g \in G | \rho_g^* \eta = \eta \,\}$$

be the stabilizer of η; it is a closed subgroup of G.

Let $G_1 = G \times_\rho E$ be the semi-direct product of G with E, for the linear representation ρ. We recall that the product in G_1 is given by the formula

$$(g,x)(h,y) = (gh, x + \rho_g y)\,,$$

where g and $h \in G$, x and $y \in E$.

With every $x \in E$, we associate the function f_x^η on G

$$f_x^\eta(g) = \bigl\langle \eta, \rho_g(x) \bigr\rangle\,.$$

Of course, the function f_x^η depends not only on $x \in E$, but also on $\eta \in E^*$; however, the parameter $\eta \in E^*$ will be considered as fixed.

We define an action ψ^η of E on T^*G by setting, for all $x \in E$ and $\xi \in T^*G$,

$$\psi_x^\eta(\xi) = \xi + df_x^\eta\bigl(q_G(\xi)\bigr)\,.$$

Finally, by composition of ψ^η with $\widehat{R}$, we obtain an action Φ_R of $G_1 = G \times_\rho E$ on T^*G, on the right, given by the formula

$$\Phi_{R\ (g,x)}(\xi) = \psi_x^\eta \circ \widehat{R}_g(\xi)\,,$$

where $(g,x) \in G \times_\rho E$ and $\xi \in T^*G$.

The main result of this section is the following theorem.

Theorem 5. *With the above assumptions and notations, the restriction to G_η of the action $\widehat{L}$ and the action Φ_R are two commuting, Hamiltonian actions on the symplectic manifold (T^*G, Ω), respectively of the Lie group G_η on the left, and of the Lie group $G_1 = G \times_\rho E$ on the right. Their momentum maps J_L^η and J_R^η take their values, respectively, in the dual $\mathcal{G}_\eta^*$ of the Lie algebra of G_η and in the dual $\mathcal{G}^* \times E^*$ of the Lie algebra $\mathcal{G} \times E$ of G_1. For each point $\xi \in T^*G$, the tangent spaces at ξ to the $\widehat{L}$-orbit of G_η and to the Φ_R-orbit of G_1 are two vector subspaces of the symplectic vector space $\bigl(T_\xi(T^*G), \Omega(\xi)\bigr)$, symplectically orthogonal to each other. When T^*G is equipped with its canonical symplectic structure, $\mathcal{G}_\eta^*$ with its minus Lie-Poisson bracket and $\mathcal{G}^* \times E^*$ with its plus Lie-Poisson bracket, the momentum maps $J_L^\eta : T^*G \to \mathcal{G}_\eta^*$ and $J_R^\eta : T^*G \to \mathcal{G}^* \times E^*$ are Ad^*-equivariant Poisson maps. Moreover, the orbits of the action $\widehat{L}$ of G_η are the level sets of J_R^η, and the orbits of the action Φ_R of G_1 are the level sets of J_L^η.*

Application to a G_η-invariant Hamiltonian system.

With the above assumptions and notations, let $H : T^*G \to \mathbb{R}$ be a smooth function, and X_H the associated Hamiltonian vector field on T^*G. Let us assume that H is invariant under the restriction of the action $\widehat{L}$ to the subgroup G_η. Since H is constant on each $\widehat{L}$-orbit of G_η in T^*G, and since these orbits are the level sets of the momentum map

$J_R^\eta : T^*G \to \mathcal{G}_1^* = \mathcal{G}^* \times E^*$, there exists a unique smooth function $\widehat{H} : \mathcal{G}^* \times E^* \to \mathbb{R}$ such that

$$H = \widehat{H} \circ J_R^\eta\,.$$

Since J_R^η is a Poisson map, the Hamiltonian vector field X_H on T^*G projects by J_R^η on $\mathcal{G}^* \times E^*$, and its projection is the Hamiltonian vector field $X_{\widehat{H}}$ associated to the Hamiltonian $\widehat{H}$, for the plus Lie-Poisson structure of $\mathcal{G}^* \times E^*$. The corresponding differential equation on $\mathcal{G}^* \times E^*$ is the Euler equation.

5. The rigid body with a fixed point

The assumptions and notations being those of Example 3 of Section 1, the Hamiltonian of the system is

$$H(\xi) = \frac{1}{2}\left\langle J_R(\xi), I^{-1} \circ J_R(\xi)\right\rangle - \left\langle {}^t(q_G\xi)F, a\right\rangle.$$

We have denoted by F the gravity force (considered as an element of the dual E^* of E), and by $a \in E$ the vector $\overrightarrow{0G}$, where 0 is the fixed point and G the center of mass of the rigid body in its reference configuration. The linear map $I : \mathcal{G} \to \mathcal{G}^*$ is the inertia operator; it is symmetric definite positive.

The rigid body with a fixed point without gravity effects.

The gravity force does not appear in the Hamiltonian either when $F = 0$ (no gravity force), or when $a = \overrightarrow{0G} = 0$ (the fixed point is the center of mass of the body). The Hamiltonian H reduces to

$$H(\xi) = \frac{1}{2}\left\langle J_R(\xi), I^{-1} \circ J_R(\xi)\right\rangle.$$

It depends only on $J_R(\xi)$; since the $\widehat{L}$-orbits are the level sets of J_R, the Hamiltonian is constant on each $\widehat{L}$-orbit, and can be written as

$$H = \widehat{H} \circ J_R\,, \quad \text{with} \quad \widehat{H}(M) = \frac{1}{2}\left\langle M, I^{-1}(M)\right\rangle, \qquad (M \in \mathcal{G}^*)\,.$$

The Euler equation is the differential equation on $\mathcal{G}^*$ defined by the Hamiltonian vector field $X_{\widehat{H}}$ associated with $\widehat{H}$, for the plus Lie-Poisson bracket on $\mathcal{G}^*$. It can be written as

$$\frac{dM(t)}{dt} = -\operatorname{ad}^*_{I^{-1}\left(M(t)\right)} M(t)\,.$$

Following Arnol'd [1,2], let us define the bilinear map $B : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ by

$$\left\langle I \circ B(X,Y), Z\right\rangle = \left\langle I(X), [Y,Z]\right\rangle, \qquad X\,,\ Y \text{ and } Z \in \mathcal{G}\,.$$

By the change of variables $\Omega = I^{-1}(M)$, the Euler equation becomes

$$\frac{d\Omega(t)}{dt} = B\bigl(\Omega(t), \Omega(t)\bigr) .$$

Remark. The variables M and Ω have a natural physical interpretation: M is the angular momentum in the reference frame of the body and Ω the angular velocity, also in the reference frame of the body.

The rigid body with a fixed point and gravity effects.

Let us now assume that neither F nor a vanish. The Hamiltonian H is no more invariant by the action $\widehat{L}$ of the full group G, but only by the restriction of that action to the subgroup G_F of G:

$$G_F = \bigl\{ g \in G \bigm| {}^t g(F) = F \bigr\} .$$

We observe that G_F is the stabilizer of F for the action of G on E^* contragredient of the natural action ρ of G on E. According to Theorem 5, the semi-direct product $G_1 = G \times_\rho E$ acts on T^*G by a Hamiltonian action on the right, Φ_R. The momentum map J_R^F of that action, which takes its values in $\mathcal{G}^* \times E^*$, can be expressed as

$$J_R^F(\xi) = \bigl(\widehat{R}_{q(\xi)^{-1}}\xi, {}^t\bigl(q_G(\xi)\bigr)F\bigr) .$$

The Hamiltonian H can be written as

$$H = \widehat{H} \circ J_R^F ,$$

where $\widehat{H} : \mathcal{G}^* \times E^* \to \mathbb{R}$ si given by

$$\widehat{H}(M, P) = \frac{1}{2} \bigl\langle M, I^{-1}(M)\bigr\rangle - \langle P, a\rangle .$$

The Euler equation is the Hamilton differential equation associated to the Hamiltonian $\widehat{H}$, the space $\mathcal{G}^* \times E^*$ being equipped with its plus Lie-Poisson bracket. For the 3-dimensional Euclidean vector space E, the scalar product yields a natural identification of E with its dual E^*; once an orientation is chosen on E, the vector product yields another identification of E with the Lie algebra $\mathcal{G}$. By combining these identifications we can consider E, E^*, $\mathcal{G}$ and $\mathcal{G}^*$ as being all the same space. The Euler equation becomes:

$$\begin{cases} \dfrac{dM}{dt} = M \times I^{-1}(M) - P \times a , \\ \dfrac{dP}{dt} = P \times I^{-1}(M) , \end{cases}$$

where $\times$ denotes the vector product in the 3-dimensional oriented, Euclidean vector space E.

6. The motion of an ideal incompressible fluid

The assumptions are now those of Example 4 of Section 1. The Lie group G is now the group $Diff(V, v)$ of volume-preserving diffeomorphisms of V. We assume that no external forces are applied to the fluid. Therefore the Hamiltonian of the system corresponds to the kinetic energy only. It is invariant under the action $\widehat{R}$ of G on T^*G, canonical lift of the action R of G on itself by right translations.

Remark. One may ask why, for the motion of a rigid body with a fixed point, without gravity effects, the Hamiltonian is invariant by the canonical lift of the action of G on itself by left translations, while for the motion of an ideal incompressible fluid, without gravity effects, the Hamiltonian is invariant by the canonical lift of the action of G on itself by right translations. The explanation is the following.

Let us begin by what is common to both systems (the ideal, incompressible fluid and the rigid body). Let $t \mapsto \varphi(t)$ be a smooth curve in the configuration space of the system. We may think about it as of a motion of the system as a function of time. Remember that for each t, $\varphi(t)$ is a map, which sends each material particle x of the system onto its position $\varphi(t)(x)$ in the physical space E when the configuration of the system is $\varphi(t)$. By choosing a particular configuration φ_0 as reference configuration, we may write $\varphi(t) = g(t) \circ \varphi_0$, where $t \mapsto g(t)$ is a smooth curve in the group G. Let h be a fixed element in G. The transforms of $t \mapsto \varphi(t)$ by the left translation L_h and by the right translation R_h are, respectively, $t \mapsto hg(t) \circ \varphi_0$ and $t \mapsto g(t)h \circ \varphi_0$. By applying this to a particular material particle x, and by taking the derivative with respect to t, we obtain the velocity, in the physical space E, of that material particle. Taking half the square of the norm of the velocity and integrating over all the material particles, with respect to the material measure, we obtain the kinetic energy.

Let us now look about what is particular to each system.

For the ideal incompressible fluid which fills a vessel V fixed in space, we see that the velocity field of the fluid, in the physical space, is exactly the same for the motions $t \mapsto g(t) \circ \varphi_0$ and $t \mapsto g(t) \circ h \circ \varphi_0$. The only difference is that the material particle which is at a given point $y \in E$ in the first motion is $x = (g(t) \circ \varphi_0)^{-1}(y)$, while in the second motion it is $(g(t)h \circ \varphi_0)^{-1}(y) = (\varphi_0^{-1} \circ h \circ \varphi_0)^{-1}(x)$. Since the part V of the physical space in which the fluid flows is fixed, and since the density of the fluid is a constant, the kinetic energy is the same in the two motions. As a consequence, the Hamiltonian of the system is invariant under the $\widehat{R}$-action.

In the contrary, since the diffeomorphism h is not, in general, a rigid displacement, the velocity fields of the fluid in physical space for the motions $t \mapsto g(t) \circ \varphi_0$ and $t \mapsto hg(t) \circ \varphi_0$ are not the same; therefore the Hamiltonian is not, in general, invariant under the $\widehat{L}$-action.

For the rigid body with a fixed point, the velocity field in the physical space is, as for the ideal fluid, the same in the motions $t \mapsto g(t) \circ \varphi_0$ and $t \mapsto g(t)h \circ \varphi_0$. But the position of the rigid body, in the physical space, is not the same in these two motions: a given point in E may be occupied by a massive particle for one of these motions, and may be empty (or occupied by a lighter particle) in the other motion. Therefore the kinetic energy is not the same in the two motions, and as a consequence, the Hamiltonian of the system is not invariant under the $\widehat{R}$-action.

In the contrary, since h is a rigid displacement, the velocity field in the physical space for the motion $t \mapsto hg(t) \circ \varphi_0$ is obtained from the velocity field in the physical space for the motion $t \mapsto g(t) \circ \varphi_0$ by the same rigid displacement, h, which maps the configuration of the rigid body in the motion $t \mapsto g(t) \circ \varphi_0$ onto its configuration in the motion $t \mapsto hg(t) \circ \varphi_0$. In other words, in a frame attached to the rigid body, the velocity fields in these two motions are the same. Therefore the kinetic energy of the system is the same in the two motions. As a consequence, when there are no gravity effects, the Hamiltonian of the system is invariant under the $\widehat{L}$-action.

Let us observe that the Hamiltonian of particular systems may have additional invariance properties. For example, the Hamiltonian of a rigid body with a revolution axis of symmetry, with a fixed point on that axis, is invariant under the restriction of the $\widehat{R}$-action to a one-parameter subgroup of $SO(E, 0)$, made of the rotations around the symmetry axis. The Hamiltonian of an ideal, incompressible fluid which fills a fixed vessel V with a revolution symmetry axis is invariant under the restriction of the $\widehat{L}$-action to a one-parameter subgroup of $Diff(V, v)$, made of the rotations around that axis; when the vessel V is a spherical cavity, the Hamiltonian of the ideal fluid is invariant under the restriction of the $\widehat{L}$-action to a 3-dimensional subgroup of $Diff(V, v)$, made of the rigid rotations around the centre of V.

The Lie algebra $\mathcal{G}$ of G can be identified with the space of divergence-free vector fields on V tangent to its boundary ∂V. However, the composition law in this Lie algebra is the *opposite* of the usual bracket of vector fields.

Let us define the pairing $(\alpha, X) \mapsto \langle \alpha, X \rangle$ of a differential 1-form α on V and a divergence-free vector field X on V, by the formula

$$\langle \alpha, X \rangle = \int_V \bigl\langle \alpha(x), X(x) \bigr\rangle \, dv(x) \, .$$

Using the fact that X is divergence-free ($\operatorname{div} X = 0$), one can prove that when α is exact (that means is the differential df of a smooth function f on V), then $\langle \alpha, X \rangle = 0$. When V is a simply connected compact subset of E with a smooth boundary ∂V, the dual space $\mathcal{G}^*$ of $\mathcal{G}$ can be identified with the quotient space $\Omega^1(V)/d\Omega^0(V)$, where $\Omega^1(V)$ is the space of differential 1-forms and $\Omega^0(V)$ the space of smooth functions on V.

We denote by ρ the volumic mass of the fluid. We assume that the fluid is homogeneous; since it is also incompressible, ρ is a constant.

Let X be an element of $\mathcal{G}$ or, in other words, a divergence-free vector field on V tangent to the boundary ∂V. Using the Euclidean structure of E, we can define the differential 1-form $X^\flat$ on V such that, for each point $x \in V$ and each vector $w \in E$,

$$\bigl\langle X^\flat(x), w \bigr\rangle = \bigl(X(x) \mid w \bigr) \, ,$$

where $(\ \mid\ )$ denotes the scalar product in E. We associate with X the differential 1-form $\rho X^\flat$, and we denote by IX the class modulo $d\Omega^0(V)$ of $\rho X^\flat$. The map $I : \mathcal{G} \to \mathcal{G}^*$ so defined is the inertia operator of the system.

The momentum map $J_L : T^*G \to \mathcal{G}^*$ is the map which associates, with each kinematic state $\xi \in T^*G$ of the system, the element IX of $\mathcal{G}^*$, where X is the velocity vector field of the fluid on V for the kinematic state ξ.

Exactly as for the motion of a rigid body around a fixed point, the Euler equation is

$$\frac{dM(t)}{dt} = -\operatorname{ad}^*_{I^{-1}(M(t))} M(t)\,.$$

With the same change of variables as in Section 5,

$$X = I^{-1}(M)\,,$$

and the same definition of the bilinear map $B : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, the Euler equation becomes

$$\frac{dX(t)}{dt} = -B\bigl(X(t), X(t)\bigr)\,.$$

The explicit expression of B, in terms of the vector product in E and of the gradient and curl operators (denoted by $\overrightarrow{\mathrm{grad}}$ and $\overrightarrow{\mathrm{curl}}$, respectively), is

$$B(X, Y) = (\overrightarrow{\mathrm{curl}}\, X) \times Y + \overrightarrow{\mathrm{grad}}\, h\,,$$

where h is the smooth function (unique up to an additive constant) such that the vector field $B(X, Y)$ is divergence-free. That function is solution of the Poisson partial differential equation

$$\Delta h + \operatorname{div}\bigl((\overrightarrow{\mathrm{curl}}\, X) \times Y\bigr) = 0\,.$$

The Euler equation is therefore

$$\frac{\partial X(t,x)}{\partial t} + \bigl(\overrightarrow{\mathrm{curl}}_x\, X(t,x)\bigr) \times X(t,x) + \overrightarrow{\mathrm{grad}}_x\, h = 0\,.$$

Remark. The smooth function h has a physical interpretation: it is related to the pressure p, the velocity X and the volumic mass ρ of the fluid by the formula

$$h = \frac{p}{\rho} + \frac{\|X\|^2}{2}\,.$$

7. The Korteweg-de Vries equation as an Euler equation

Let $\mathcal{G}$ be the Lie algebra of smooth vector fields on the circle S^1, with the bracket opposite to the usual bracket of vector fields,

$$\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right] = \big(f'(x)g(x) - g'(x)f(x)\big)\frac{\partial}{\partial x}\,,$$

where x is the angular coordinate on S^1 (defined modulo 2π). The components f and g of the two vector fields on S^1 are considered as 2π-periodic, smooth functions on $\mathbb{R}$.

Let $c : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ be the bilinear map

$$c\left(f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right) = \int_{S^1} f'(x)g''(x)\,dx\,.$$

An integration by parts shows that c is skew-symmetric:

$$c\left(f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right) = -c\left(g(x)\frac{\partial}{\partial x}, f(x)\frac{\partial}{\partial x}\right)\,.$$

The map c satisfies the identity

$$c\left(\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right], h(x)\frac{\partial}{\partial x}\right) + \text{ cyclic sum } = 0\,.$$

The map c is called the *Gelfand-Fuchs cocycle* of the Lie algebra $\mathcal{G}$.

Let $\mathcal{G}_1 = \mathcal{G} \times \mathbb{R}$, equipped with the bracket

$$\left[\left(f(x)\frac{\partial}{\partial x}, a\right), \left(g(x)\frac{\partial}{\partial x}, b\right)\right] = \left(\left[f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right], c\left(f(x)\frac{\partial}{\partial x}, g(x)\frac{\partial}{\partial x}\right)\right)\,.$$

With that bracket, $\mathcal{G}_1$ is an infinite-dimensional Lie algebra, called the *Virasoro algebra*.

There exists an infinite-dimensional Lie group G_1 whose Lie algebra is the Virasoro algebra. It is called the *Virasoro-Bott group*, and it is the semi-direct product $Diff(S^1)\times\mathbb{R}$ of the goup of diffeomorphisms of the circle with $\mathbb{R}$, with the composition law

$$(\varphi, a)(\psi, b) = \left(\varphi \circ \psi, a + \int_{S^1} \ln(\varphi\circ\psi)'\,d(\ln\psi')\right)\,.$$

The dual space $\mathcal{G}_1^*$ of $\mathcal{G}_1$ is the product $\mathcal{G}^* \times \mathbb{R}$, where $\mathcal{G}^*$ is the space of differentiable 1-forms on S^1. The pairing of $\mathcal{G}_1^*$ with $\mathcal{G}_1$ is given by the formula

$$\left\langle \big(g(x)\,dx, b\big), \left(f(x)\frac{\partial}{\partial x}, a\right)\right\rangle = \int_{S^1} g(x)f(x)\,dx + ab\,.$$

Let $I : \mathcal{G}_1 \to \mathcal{G}_1^*$ be the linear map

$$I\left(f(x)\frac{\partial}{\partial x}, a\right) = \big(f(x)\,dx, a\big)\,,$$

and let $\widehat{H} : \mathcal{G}^* \to \mathbb{R}$ be the function

$$
\begin{aligned}
\widehat{H}\big(g(x)\,dx, a\big) &= \frac{1}{2}\left\langle \big(g(x)\,dx, a\big), I^{-1}\big(g(x)\,dx, a\big)\right\rangle \\
&= \frac{1}{2}\left(\int_{S^1} \big(g(x)\big)^2\,dx + a^2\right)
\end{aligned}
.
$$

Let H be the Hamiltonian on the cotangent bundle T^*G_1 of the Virasoro-Bott group G_1, invariant under the $\widehat{R}$-action of G_1, given by

$$
H = \widehat{H} \circ J_L\,,
$$

where J_L is the momentum map of the $\widehat{L}$-action of G_1. The corresponding Euler equation on $\mathcal{G}_1^*$, given by the same formulas as in the previous sections, is

$$
\frac{\partial}{\partial t}\big(g(x,t)\,dx, a(t)\big) = -\operatorname{ad}^*_{I^{-1}\left(g(x,t)\,dx, a(t)\right)}\big(g(x,t)\,dx, a(t)\big)\,.
$$

By using the expression of the bracket in $\mathcal{G}_1$, and after several integrations by parts, the Euler equation can be written as

$$
\frac{\partial}{\partial t}\big(g(x,t)\,dx, a(t)\big) = \left(b(t)\,\frac{\partial^3 g(x,t)}{\partial x^3} + 3g(x,t)\,\frac{\partial g(x,t)}{\partial x}, 0\right)\,.
$$

The second component of that equation is

$$
\frac{\partial b(t)}{\partial t} = 0\,,
$$

therefore b is a constant. The first component of the above equation becomes:

$$
\frac{\partial g(x,t)}{\partial t} = b\,\frac{\partial^3 g(x,t)}{\partial x^3} + 3g(x,t)\,\frac{\partial g(x,t)}{\partial x}\,.
$$

That equation is the famous Korteweg-de Vries equation on the circle.

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