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Kinematic and geometric constraints, servomechanisms and control of mechanical systems

En hommage amical au Professeur Tulczyjew,
dont les travaux sont une source constante d'inspiration.

1. Introduction

The theory of mechanical systems with kinematic constraints goes back to the last century, with important contributions by Hertz (1894), Ferrers (1871), Neumann (1888), Vierkandt (1892), Chaplygin (1897). See the references in the treatise by E. T. Whittaker [29], and in J. Herman's thesis [15]. Several recent papers [3–6, 8, 9, 12–16, 19, 20, 24–26, 28] show a strong renewal of interest in that theory, in relation with new developments in control theory and sub-riemannian geometry.

In this paper, for simplicity, we will consider only time-independent constraints. We will try to present our own viewpoint, and to advocate the idea that in general cases, the knowledge of the set of all possible kinematic states of the constrained system is not sufficent to determine completely the dynamics: one must know in addition the space of possible values of the constraint forces. This simple fact, which was clearly seen for example by P. Dazord [9], seems to have been frequently overlooked in the recent literature (even by the present author in [18]), probably because when the kinematic constraints to which the system is submitted are linear (or, more generally, affine) in the velocities, perfect and passive, the so called d'Alembert's principle allows the determination of the set of possible values of the constraint forces when the manifold of possible kinematic states is known. By using servomechanisms, one can easily realize systems with constraints which are no more affine in the velocities. Simple examples will be given which show that for such constraints, d'Alembert's principle (or its natural generalization, Chetaev's rule) can no more be used to determine the space of possible values of the constraint forces.

The total energy of a constrained mechanical system is not in general a first integral of the motion. Similarly, when a Lie group acts on the system by a Hamiltonian action, the momentum map may not remain constant during the motion (contrary to what happens for unconstrained systems). We will indicate sufficient conditions under which the total energy of the system or the momentum map are first integrals of the motion.

Finally we will show by an example that kinematic constraints, realized by means of servomechanisms, can be used to control the motion of a mechanical system, and more specifically, to stabilize an otherwise unstable equilibrium.

2. Mechanical systems with additional forces

2.1. Lagrange's formalism.

We consider a mechanial system, for which the set of all possible positions of its parts, at a given time, is a smooth manifold Q. That manifold will be called the *configuration space* of the system. The set of all possible positions and velocities of the parts of the system, at a given time, is then the tangent bundle TQ, which will be called the space of *kinematic states* of te system.

We will assume that the dynamical properties of the system are mathematically described by a smooth function $L : TQ \to \mathbf{R}$, called the *Lagrangian* of the system. That function determines a map $\Delta(L)$, called the *Lagrange differential* of L [21, 23], defined on the set $J^2(\mathbf{R}, 0, Q)$ of second order jets of $\mathbf{R}$ into Q at the origin, with values in the cotangent bundle T^*Q, fibered over Q. That map may be defined in an intrinsic way [21, 23]. Let us simply recall its expression in local coordinates. We use a chart of Q in which the local coordiantes are $(x^i,\ 1 \leq i \leq n)$, and we denote by (x^i, v^i) and (x^i, p_i) the local coordiantes in the associated charts of TQ and T^*Q. Let $c : \mathbf{R} \to Q$ be a parametrized C^2 curve in Q. We denote by $\bigl(x^i(t), v^i(t)\bigr)$ the local coordinates of $\dfrac{dc(t)}{dt}$. Then

$$\Delta(L)\bigl(j^2c(t_0)\bigr) = \sum_{i=1}^{n} \left(\frac{d}{dt}\left(\frac{\partial L\bigl(x(t), v(t)\bigr)}{\partial v^i}\right) - \frac{\partial L\bigl(x(t), v(t)\bigr)}{\partial x^i} \right)\Bigg|_{t=t_0} \frac{\partial}{\partial p_i}\,.$$

We will assume that some additional forces, not already accounted for by the Lagrangian L, are acting on the system. These forces will be specified in the next section: they will be the constraint forces. When the system's configuration is a point $x \in Q$, these additional forces are mathematically described by an element f (in general unknown) of the cotangent space T^*_xQ. The equations of motion of the system, in Lagrange's formalism, can be written in coordinate-free form, as

$$\Delta(L)\bigl(j^2c(t)\bigr) = f\,, \tag{1}$$

or, in local coordinaters,

$$\frac{d}{dt}\left(\frac{\partial L(x, v)}{\partial v^i}\right) - \frac{\partial L(x, v)}{\partial x^i} = f_i\,, \qquad 1 \leq i \leq n\,. \tag{2}$$

2.2. Hamilton's formalism.

The Lagrangian L determines, in a unique way, a smooth map $\mathcal{L} : TQ \to T^*Q$, fibered over Q, called the *Legendre transformation*. We refer to Tulczyjew [22] for an intrinsic very general description of that transformation, and recall here that if we use, as above, local coordinates (x^i, v^i) on TQ and (x^i, p_i) on T^*Q, the Legendre transformation is expressed as

$$\mathcal{L} : (x^i,\ v^i) \mapsto \left(x^i,\ p_i = \frac{\partial L(x, v)}{\partial v^i} \right)\,.$$

We will assume in the following that the Lagrangian L is regular, *i.e.*, that the Legendre transformation $\mathcal{L}$ is a diffeomorphism of TQ onto T^*Q. The cotangent bundle T^*Q will be called the *phase space* of the mechanical system, and points in T^*Q will be called *dynamical states* of that system.

A motion, given by a parametrized smooth curve $t \mapsto c(t)$ in Q, will be represented, in Hamilton's formalism, by the parametrized curve $t \mapsto \widetilde{c}(t) = \mathcal{L}\left(\dfrac{dc(t)}{dt}\right)$ in T^*Q. We introduce the Hamiltonian $H : T^*Q \to \mathbf{R}$, given by $H = \bigl(i(Z)L - L\bigr) \circ \mathcal{L}^{-1}$, where Z is the Liouville vector field on TQ. We introduce also the Liouville 1-form α, the canonical symplectic 2-form $\Omega = d\alpha$ on T^*Q, and the Hamiltonian vector field X_H associated with H, defined by $i(X_H)\Omega = -dH$.

A final ingredient is needed in order to write down the equations of motion in Hamilton's formalism: an intrinsic way of introducing the additional forces. For that purpose we define a map $\lambda : T^*Q \times_Q T^*Q \to TT^*Q$, as follows: for $x \in Q$, ξ and $\eta \in T^*_xQ$, $\lambda(\xi, \eta)$ is the vector, element of $T_\xi(T^*Q)$, tangent at ξ to the fibre T^*_xQ, and equal to η (the tangent space at ξ to the vector space T^*_xQ being identified with that vector space). We observe that $\lambda(\xi, \eta)$ is vertical: that means that if we denote by $q : T^*Q \to Q$ the canonical projection and $Tq : TT^*Q \to TQ$ its canonical lift to vectors, then $Tq\bigl(\lambda(\xi, \eta)\bigr) = 0$. Let us assume that the dynamical state of the system is a point $\xi \in T^*Q$, the corresponding configuration being $x = q(\xi) \in Q$, and that the additional forces are represented, in Lagrange's formalism, by $f \in T^*_xQ$. Then these additional forces will be represented, in Hamilton's formalism, by the vertical vector $\lambda(\xi, f)$, tangent to T^*Q at point ξ.

In Hamilton's formalism, the equations of motion are the well known Hamilton's equations, modified by introduction of the additional forces. They can be written, in coordinate-free form, as

$$\frac{d\widetilde{c}(t)}{dt} = X_H\bigl(\widetilde{c}(t)\bigr) + \lambda\bigl(\widetilde{c}(t), f\bigr)\,, \tag{3}$$

or, in local coordinates,

$$\begin{cases} \dfrac{dx^i}{dt} = \dfrac{\partial H(x,p)}{\partial p_i}\,, \\[2ex] \dfrac{dp_i}{dt} = -\dfrac{\partial H(x,p)}{\partial x^i} + f_i\,, \end{cases} \qquad 1 \leq i \leq n\,. \tag{4}$$

3. Mechanical systems with constraints

The mechanical system under consideration is sait to be *constrained*, or *submitted to kinematic constraints*, when the set of its possible kinematic states is a subset C of TQ, rather that the whole tangent bundle TQ. Such a limitation of the set of possible kinematic states is generally due to interactions of some parts of the system, either between themselves, or with external objects, not already accounted for by the definition of the configuration space Q. These interactions give rise to additional forces, not already accounted for by the Lagrangian L of the system. These forces, which will be called the *constraint forces*, are the additional forces introduced in the previous section, which were left unspecified up to now.

We shall assume in the following that the subset C of possible kinematic states is a smooth submanifold of TQ, called the *constraint submanifold*. Since the Legendre transformation $\mathcal{L}$ is a diffeomerphism of TQ onto T^*Q, the image $\mathcal{L}(C)$ of the constraint submanifold by the Legendre transformation is a smooth submanifold D of T^*Q, which will be called the *Hamiltonian constraint submanifold*. Let us observe that some authors, for example Weber [28], define a Hamiltonian constrained system by means of a distribution on the phase space T^*Q, instead of by means of a submanifold. We think that for applications to mechanical systems, the use of a submanifold is much more natural.

In Lagrange's formalism, the motion of the mechanical system must satisfy Lagrange's equations (1) and the constraint equation

$$\frac{dc(t)}{dt} \in C \quad \text{for all } t\,. \tag{5}$$

Equivalently, in Hamilton's formalism, the motion of the mechanical system must satisfy Hamilton's equations (3), and the constraint equation

$$\widetilde{c}(t) \in D \quad \text{for all } t\,. \tag{6}$$

When the constraint force f, which appears in Equations (1) or (3), is considered as unknown, the system made by (1) and (5) (or by (3) and (6)) is underdetermined. The additional equations which are needed must express physical properties of the constraint. As we shall see in the following examples, these additional equations generally restrict the space of possible values of the constraint force.

3.1. Classical kinematic constraints. A kinematic constraint will be called *classical* when it is linear, or more generally affine, in the velocities, *i.e.*, when the corresponding constraint submanifold C is an affine sub-bundle of a tangent bundle TQ_1, where Q_1 is a smooth submanifold of the configuration manifold Q.

Simplest examples of classical constraints are the *geometric constraints*, in which the restriction bears upon the set of possible configurations of the system, rather than on the set of possible kinematic states: the set of possible configurations is a smooth submanifold Q_1 of Q, and the constrtaint submanifold is the tangent bundle TQ_1, considered as a submanifold of TQ.

In more general classical kinematic constraints, the restriction bears upon possible velocities of parts of the system, as well as on possible configurations. Such constraints are encountered, in particular, in systems made of several rigid bodies, some of which are rolling without slipping on each other, or on other rigid bodies which do not belong to the system and whose motions are known and stationary.

EXAMPLE 1. Let us consider a system made of a rigid three-dimensional heavy body. Let E be the physical space, G the group of Euclidean dispalcements of E, and P_0 a particular position of the rigid body in E. By associating with any position P of the body, the unique $g \in G$ such that $P = gP_0$, we can identify the configuration space Q of the system with the group G. Choosing an origin O in E allows us to consider E as an Euclidean three-dimensional vector space and to identify G with the semi-direct product $E \times \mathbf{SO}(E)$, an element (a, g) of that semi-direct product corresponding to the mapping of E into itself $x \mapsto a + gx$. We identify the Lie algebra $\mathcal{G}$ of G with $E \times \mathbf{so}(E)$, and the tangent bundle TG with $G \times \mathcal{G} = E \times \mathbf{SO}(E) \times E \times \mathbf{so}(E)$, by left translations.

Let us now assume that the rigid body is bounded by a smooth strictly convex surface and that it is supported by a fixed solid horizontal plane F. Moreover, let us assume that during its motion, the body remains in contact with the horizontal plane and that it rolls without slipping on it. We choose the origin O in the horizontal plane F, and the unit vector e_3 vertical and directed upwards. We denote by Σ the surface of the rigid body when it is in its reference position P_0, and by $\Gamma : S^2 \to \Sigma$

the inverse of the Gauss map of Σ. Then the constraint submanifold is the set of kinematic states (a, g, v, X) which satisfy

$$(a|e_3) = \big(\Gamma \circ g^{-1}(-e_3) \bigm| g^{-1}(-e_3)\big)\,, \tag{7}$$

$$v + X \circ \Gamma \circ g^{-1}(-e_3) = 0\,, \tag{8}$$

where $(\ |\ )$ denotes the Euclidean scalar product in E (we refer to [18] for more details). The set of $(a, g) \in G$ which satisfy Equation (7) is a submanifold Q_1 of G which may be identified with $F \times \mathbf{SO}(E)$. Then Equation (8) shows that the constraint submanifold C is a (non integrable) vector sub-bundle of TQ_1.

Example 2. Let us now assume that the plane on which the body rests rotates around the vertical axis through the origin, at a given constant angular velocity ω. The constraint submanifold C is now the set of kinematic states (a, g, v, X) which satisfy Equation (7) and

$$g\big(v + X \circ \Gamma \circ g^{-1}(-e_3)\big) = \omega\big(a + g \circ \Gamma \circ g^{-1}(-e_3)\big)\,, \tag{9}$$

instead of (8). Clearly, C is now an affine sub-bundle of TQ_1, whose associated vector sub-bundle is the sub-bundle obtained previously as constraint submanifold when the plane F was assumed to be fixed, defined by Equation (7).

3.2. Perfect classical kinematic constraints.

We now come to the description of constraint forces for a classical kinematic constraint. Let C be the constraint submanifold. According to the definition given in the previous section, C is an affine sub-bundle of a tangent bundle TQ_1, where Q_1 is a smooth submanifold of Q. For any $x \in Q_1$, we denote by C_x the fibre $C \cap T_xQ$ of C over x; therefore C_x is an affine subspace of T_xQ_1, which is itself a vector subspace of T_xQ, so we can consider C_x as an affine subspace of T_xQ. We shall denote by $\overrightarrow{C}_x$ the vector subspace of T_xQ associated with C_x, and by $(\overrightarrow{C}_x)^0$ its annihilator, *i.e.*, the vector subspace of elements η in T^*_xQ such that, for all $v \in \overrightarrow{C}_x$, $\langle \eta, v\rangle = 0$. When x runs over Q_1, we obtain a vector sub-bundle $\overrightarrow{C}$ of $T_{Q_1}Q$, whose fibre at x is $\overrightarrow{C}_x$, called the bundle of *admissible infinitesimal virtual displacements*. Its annihilator $(\overrightarrow{C})^0$ is the vector sub-bundle of $T^*_{Q_1}Q$ whose fibre at x is $(\overrightarrow{C}_x)^0$.

According to the so called d'Alembert's principle, the classical constraint is said to be *perfect* if the infinitesimal work of the constraint force vanishes for any admissible infinitesimal virtual displacement. This amounts to say that the constraint force takes its value in $(\overrightarrow{C})^0$.

D'Alembert's principle has the following nice mathematical property: for a mechanical system with perfect classical kinematic constraints, the restriction of the space of constraint forces obtained by application of that principle, added to Equations (1) and (5) in Lagrange's formalism (or Equations (3) and (6) in Hamilton's formalism), leads to a well behaved system of differential equations. We shall illustrate that property for the simple example of a geometric constraint, where $C = TQ_1$. In that case, an additional simplification occurs: the geometric constraint can be eliminated by using Q_1 instead of Q as configuration space and $L_1 = L\big|_{TQ_1}$ instead of L as Lagrangian. Let us indeed use a chart of Q adapted to the submanifold Q_1, in which the local coordinates $(x^i,\ 1 \leq i \leq n)$ are such that Q_1 is locally defined by the equations $x^{p+1} = 0,\ \dots,\ x^n = 0$. The equations of motion of the system, in

Lagrange's formalism, are

$$\frac{d}{dt}\left(\frac{\partial L(x,v)}{\partial v^i}\right) - \frac{\partial L(x,v)}{\partial x^i} = 0\,, \qquad 1 \le i \le p\,, \tag{9}$$

$$\frac{d}{dt}\left(\frac{\partial L(x,v)}{\partial v^j}\right) - \frac{\partial L(x,v)}{\partial x^j} = f_j\,, \qquad p+1 \le j \le n\,, \tag{10}$$

$$x^j = 0\,, \qquad p+1 \le j \le n\,. \tag{11}$$

Denoting by $y = (y^1, \ldots, y^p)$ the local coordinates on Q_1 and by $(y, w) = (y^1, \ldots, y^p, w^1, \ldots, w^p)$ the associated local coordinates on TQ_1, we see that Equations (9) and (11) imply

$$\frac{d}{dt}\left(\frac{\partial L_1(y,w)}{\partial w^i}\right) - \frac{\partial L_1(y,w)}{\partial y^i} = 0\,, \qquad 1 \le i \le p\,,$$

which are simply the Euler-Lagrange's equations for the Lagrangian L_1 on TQ_1. Equations (10) need not be used to determine the motion; they just yield the value of the constraint force.

That elimination of the geometric constraint can be observed also in Hamilton's formalism: assuming that both L and its restriction L_1 are regular, and denoting by $\mathcal{L}_1 : TQ_1 \to T^*Q_1$ the Legendre transformation determined by L_1, we see that $\mathcal{L} \circ \mathcal{L}_1^{-1}$ is a diffeomorphism of T^*Q_1 onto D, and that the equations of motion on D imply the usual Hamilton's equations, for the unconstrained Hamiltonian system on T^*Q_1 whose Hamiltonian is $H_1 = H \circ \mathcal{L} \circ \mathcal{L}_1^{-1}$.

3.3. Non classical kinematic constraints.

Mechanical systems with kinematic constraints nonlinear in the velocities were considered very early (P. Appell [1, 2], 1911). A very abundant literature, extending up to now, deals with various methods to obtain their equations of motion: application of Gauss' principle of least curvature, conditional Hamilton's variational principle, Hamilton-Jacobi equation, ... The results obtained are sometimes contradictory, and it is generally recognized now (thanks to Faddeev and Vershik, among others) that these various methods do not lead to equivalent equations. In our opinion, the main difficulty encountered with such systems is the following. We have seen in Section 2 that when the constraint forces are considered as unknown, the system made by Lagrange's equations with additional forces (1) and by the constraint equations (5) (or, in Hamilton's formalism, by Hamilton's equations with additional forces (3) and the constraint equations (6)) is underdetermined. One may think that some physical properties of the constraints should impose restrictions to the set of possible values of the constraint forces, in such a way that the system of equations will become well behaved. It is exactly what occurs for classical perfect constraints, with the restrictions imposed by d'Alembert's principle. But it is not at all clear what should be defined as a perfect constraint, when that constraint is non classical (*i.e.*, neither linear nor affine in the velocities).

Around 1930, Chetaev [7] proposed a rule to restrict the set of possible values of the constraint forces. Non classical constraints which obey that rule (discussed, for example, by Pironneau [20]) are known, at least in the Russian literature, as constraints of Chetaev's type; let us observe, however, that equations written earlier by Appell in [1] are in fact precisely those one would obtain by application of Chetaev's rule! Expressed in an intrinsic form, that rule is the following. Let C be the constraint submanifold, $v \in C$ a kinematic state, and $x = p(v)$ the corresponding configuration (we have denoted by $p : TQ \to Q$ the canonical projection). The fibre $C_x = C \cap T_xQ$ is assumed smooth enough so that there exists a tangent space T_vC_x to it at point v. Since T_xQ is a vector space, T_vC_x can be considered as one of its vector subspaces. When the kinematic state of the system is v, Chetaev's rule states that the space of possible values of the constraint force is the annihilator of T_vC_x. Clearly, when the constraint is classical, C_x is an affine subspace of T_xQ and Chetaev's rule is equivalent to d'Alembert's principle. That rule was introduced more recently in a different setting by Vershik and Faddeev [24, 25, 26], and used in [18] as a definition of perfect non classical constraints, where we have shown that under some rather mild assumptions, it leads to a well behaved system of differential equations for the motion of the system.

Application of Chetaev's rule may lead to wrong results. This can be seen on the famous example, first considered by Appell [1], of a heavy material point moving in space, with the kinematic constraint

$$v_x^2 + v_y^2 = a^2 v_z^2 \,,$$

where a is a constant, and v_x, v_y and v_z the components of the velocity of the moving point in a Cartesian frame with a vertical z axis. Applying Chetaev's rule, we can easily see that the trajectories of the moving point are straight lines. However, Appell [2] described a way in which that kinematic constraint can be realized in practice, by using a wheel rolling without sliding on a horizontal plane, a thread wounded on a smaller coaxial wheel, and skates sliding without friction on the plane. Appell's machine is described also in [20]. A careful analysis of that machine shows that its configuration space is at least 4-dimensional (while the configuration space of the material point alone, when ignoring the accessories used for the practical realization of the constraint, is 3-dimensional), and that the motion of the moving point submitted to that kinematic constraint by means of Appell's machine is a spiral rather than a straight line. Delassus [10, 11] discussed further these results, by considering a mechanical system S coupled with another mechanical system S_1, the total system being submitted to classical constraints (linear in the velocities), in such a way that when one eliminates the coordinates of S_1 and their time derivatives, one obtains for S alone a non classical kinematic constraint. He proves that when the masses of all the parts of S_1 vanish, the equations of motion of S do not converge towards the equations which would be obtained by applying Chetaev's rule to the non classical constraint of S, considered alone. Of course, at that time (1911), Chetaev's rule was not yet called by that name.

Practical realization of non classical kinematic constraints by means of wheels, skates, and similar mechanical components is rather difficult (although not impossible, as shown by Appell's example). By using servomechanisms, such a realization becomes much easier, at least conceptually. Let us look at simple examples.

EXAMPLE 3. This example is an idealization of the game in which one tries to keep a straight rod in equilibrium on the tip of one finger. For simplicity we assune that the rod remains in a fixed vertical plane and that the tip of the finger which supports it can move only along a straight horizontal line contained in that plane. We take that line as axis of coordinates Ox, the other axis Oz being vertical. The rod is free to rotate around its point of contact with the Ox axis. The configuration space is therefore $Q = \mathbf{R} \times S^1$, with coordinates (x, θ), where $x \in \mathbf{R}$ is the abscissa of the point of contact of the rod with the horizontal axis, and $\theta \in S^1$ the angle made by the rod with that axis. We assume that the strategy used by the player to keep the rod in equilibrium amounts to impose the value of $\dfrac{dx}{dt}$ as a function of x, θ and $\dfrac{d\theta}{dt}$. Such a strategy could be realized also with a servomechanism, instead of a human player. The constraint submanifold $C \subset TQ$ is therefore

$$C = \left\{ (x, \theta, \dot{x}, \dot{\theta}) \mid \dot{x} = f(x, \theta, \dot{\theta}) \right\},$$

where f is a known smooth function.

If the rotation of the rod around its point of contact with Ox is frictionless, the constraint force must belong to the annihilator of the vector sub-bundle of TQ generated by the vector field $\dfrac{\partial}{\partial \theta}$. Clearly, that result has nothing to do with Chetaev's rule.

EXAMPLE 5. Let us change slightly the previous example. We assume now that the servomechanism acts upon the angle θ of the rod with the horizontal axis, and that the point of contact of the rod with that axis can slide freely without friction. For simplicity we assume that the relation which links x, θ, $\dfrac{dx}{dt}$ and $\dfrac{d\theta}{dt}$ in the previous example can be solved with respect to $\dfrac{d\theta}{dt}$, as well as with respect to $\dfrac{dx}{dt}$. Then the constraint submanifold C will be the same as that in the previous example. But the constraint force must now belong to the annihilator of the vector sub-bundle of TQ generated by the vector field $\dfrac{\partial}{\partial x}$.

These two examples show that for non classical constraints realized by means of servomechnisms, Chetaev's rule cannot be used, and that the set of possible values of the constraint force is not determined by the constraint submanifold.

3.4. A general setting for constrained Hamiltonian systems.

For the treatment of mechanical systems with constraints, we have already introduced several ingredients: the configuration space Q, the space of kinematic states TQ, a submanifold C of TQ called the constraint submanifold, the Lagrangian $L : TQ \to \mathbf{R}$. Assuming that the Lagrangian is regular, we may use equivalently Hamilton's formalism, in which the ingredients are the phase space T^*Q, the Hamiltonian constraint submanifold $D = \mathcal{L}(C) \subset T^*Q$, and the Hamiltonian $H : T^*Q \to \mathbf{R}$. For simplicity, we will consider in the following only Hamilton's formalism. In order to be able to deal with arbitrary constraints, including those realized by means of servomechanisms, we propose to introduce an additional ingredient: a vector sub-bundle W of $T_D(T^*Q)$, contained in the vertical sub-bundle $\ker(Tq)$, which will be the space of possible values of the constraint forces (in Hamilton's formalism).

More generally, to deal with systems obtained by reduction, we proposed [18] to replace the cotangent bundle T^*Q by a Poisson manifold (P, Λ) (see [17] and [27] for the definition nand properties of Poisson manifolds). The Hamiltonian constraint submanifold is then a submanifold D of P and the set of possible values of the constraint forces a vector sub-bundle W of $T_D P$. The multiplet (P, Λ, H, D, W) will be called a *constrained Hamiltonian system*. Observe that our definition of that concept is not equivalent with that of Weber [28]. The constrained Hamiltonian system (P, Λ, H, D, W) will be called *regular* if $TD \cap W = \{0\}$ (the zero sub-bundle of $T_D P$), and if the restriction to D of the Hamiltonian vector field X_H is a section of the direct sum $TD \oplus W$. A similar regularity condition was introduced by Vershik and Faddeev [24, 25] under a slightly different form. When it is satisfied, we have a well behaved system of differential equations as equations of motion, since the restriction to D of the Hamiltonian vector field X_H splits into a sum

$$X_H\big|_D = X_D + X_W \tag{12}$$

of a vector fild X_D tangent to D (whose integral curves are the motions of the mechanical system), and a section X_W of W (which is the opposite of the set of values of the constraint forces). A sufficient condition of regularity is given in [18].

4. Energy and momentum

In a constrained Hamiltonian system, the Hamiltonian (that means, the total energy of the system) may not remain constant during the motion. This fact is easily understood when the constraint is realized by means of a servomechanism, since that mechanism can give energy to (or withdraw energy from) the mechanical system. For a regular constrained Hamiltonian system, one can easily see that the Hamiltonian H remains constant during every motion of the system if and only if for any kinematic state $v \in C$,

$$\langle X_W\bigl(\mathcal{L}(v), v\rangle = 0\,,$$

where $\mathcal{L}$ is the Legendre transformation and X_W the section of W defined in Equation (12). A sufficient condition under which that property holds true is when for each kinematic state $v \in C$, the vector subspace $W\bigl(\mathcal{L}(v)\bigr)$ is contained in the annihilator of v.

Let us assume that a symmetry group G acts on a constrained Hamiltonian system by a Hamiltonian action, leaving invariant all elements of its structure: the Poisson tensor Λ, the Hamiltonian H, the Hamiltonian constraint submanifold D and the bundle W. Let us assume that there exists a momentum map J of that action. Then J may not remain constant during the motion. A sufficient condition under which the momentum map J remains constant during every motion of the system is $W \subset \ker(TJ)$. When the phase space is a symplectic manifold, $\ker(TJ)$ is the symplectic orthogonal of the bundle tangent to the group orbits.

Example 5. We return to Example 1 in section 3.1, where the mechanical system is a convex solid body which rolls without slipping on a horizontal plane F. The group $F \times S^1$ acts on the system by a Hamiltonian action. The corresponding momentum map has as components the horizontal component of the total momentum of the body, and the vertical component of the angular momentum of the body with respect to the origin. It does not remain constant in time.

When the convex body is a dynamically homogeneous sphere, there is another symmetry group, $\mathbf{SO}(3)$, which acts on the system by a Hamiltonian action. The restriction of that action to the subgroup S^1 of rotations around the vertical axis through the center of the sphere satisfies the condition under which the corresponding momentum map remains constant. That momentum map is the angular momentum of the body with respect to that axis.

4. Control of mechanical system by means of a kinematic constraint

We return to Example 3 of section 3.3. Calculations show that the equilibrium position where the rod is vertical and rests on its lower end, which is unstable when no kinematic constraint is applied, becomes stable with a kinematic constraint in which $\frac{dx}{dt}$ is imposed as a suitable function of θ and $\frac{d\theta}{dt}$. For example, we may choose

$$\frac{m(I+ml^2\cos^2\theta)}{I+ml^2}\,\frac{dx}{dt} = a\left(\theta-\frac{\pi}{2}\right) - \left(\frac{I+ml^2\cos^2\theta}{l(I+ml^2)}+\beta\right)p_\theta\,,$$

where m is the mass of the rod, l the distance between the point of the rod on the Ox axis and its centre of mass, I its moment of inertia with respect to its centre of mass, a and β two constants wit $a<0$ and $\beta>0$, and

$$p_\theta = -ml\sin\theta\frac{dx}{dt} + (I+ml^2)\frac{d\theta}{dt}\,.$$

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