The inception of symplectic geometry: the works of Lagrange and Poisson during the years 1808–1810

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Abstract

We analyse articles by Lagrange and Poisson written two hundred years ago which are the foundation of present-day symplectic and Poisson geometry.

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1 Introduction

Some two hundred years ago, Joseph-Louis Lagrange (1736–1813) and Siméon Denis Poisson (1781–1840) published articles which contained the first appearance of symplectic and Poisson structures, and of related concepts.[1] The word *symplectic*, used for the first time with its modern mathematical meaning by Hermann Weyl (1885–1955) in his book *The classical groups*, first published in 1939 [29], derives from a Greek word meaning *complex*. Weyl used it because the word *complex*, whose origin is Latin, had already a different meaning in mathematics. However, the concept of a *symplectic structure* is much older than the word *symplectic* since it appeared in the works of Lagrange, first in his 1808 paper [15] about the slow variations of the orbital elements of the planets in the solar system, then again a few months later in [16] as a fundamental ingredient in the mathematical formulation of any problem in mechanics.

Most modern textbooks present, as a first and fundamental example of a symplectic structure, the structure determined on the cotangent bundle of a smooth manifold by the exterior derivative of its canonical 1-form. It is in a slightly different context that the concept of a symplectic structure first appeared in the work of Lagrange, since it

[1] I recall the memory of Nikolay Nekhoroshev, who passed away on 18 October 2008. He obtained many important results in the theory of dynamical systems, in particular some fundamental results on the stability in finite time intervals of almost completely integrable Hamiltonian systems, which may be seen as a continuation of the works of Lagrange and Poisson discussed in the present paper.

is on the *manifold of motions* of a mechanical system, rather than on the *phase space* of that system, *i.e.*, the cotangent bundle of its configuration space, that he defined a symplectic structure. There are several reasons for considering Lagrange's point of view as more appropriate than the current one. For example, in most modern textbooks, the conservation of the symplectic 2-form under the flow of a Hamiltonian vector field is presented as an important theorem in symplectic geometry, while this result had already been known to Lagrange, who considered it to be a direct consequence of the existence, on the manifold of motions of the system, of a well defined, time-independent symplectic structure.

While Lagrange introduced the concept of a symplectic structure, Poisson defined the composition law today called the *Poisson bracket*. From June 1808 to February 1810, five papers were published, two by Poisson and three by Lagrange, each paper improving on the results of the preceding one. I will present a reading of these works in the language of today's mathematicians, and I shall use modern notations and concepts when they can help to better understand Lagrange's and Poisson's ideas. My point of view will be that of a working mathematician, rather than that of a specialist in the history of mathematics.

Section 2 describes two closely related concepts: the *flow* and the *manifold of motions* of a smooth dynamical system. Section 3 describes the problem of the quantitative determination of the motion of the planets in the solar system, which was the main motivation for the work on dynamics by Lagrange and Poisson. Section 4 presents Lagrange's and Poisson's works on the method of varying constants. Finally, Section 5 offers a modern account of this method, using today's notations and concepts.

2 Flow and manifold of motions

Let us consider a smooth dynamical system, *i.e.*, an ordinary differential equation, on a smooth manifold M,

$$\frac{d\varphi(t)}{dt} = X\bigl(t,\varphi(t)\bigr)\,.$$

On the right-hand side, $X : \Omega \to TM$ is a smooth, *i.e.*, C^∞, vector field, defined on an open subset Ω of $\mathbb{R}\times M$. The map φ, defined on an open interval of $\mathbb{R}$, with values in M, is said to be a *solution* of this equation. For greater generality, we consider that, for $t\in\mathbb{R}$ and $x\in M$ such that $(t,x)\in\Omega$, $X(t,x)$, which is an element in T_xM, may depend on t. We then say that X is a time-dependent vector field. Of course, a smooth vector field defined on M in the usual sense can be considered as a time-dependent vector field defined on $\mathbb{R}\times M$ which, for each $x\in M$, is constant on the line $\mathbb{R}\times\{x\}$.

2.1 The flow of a smooth differential equation

The *flow* of a differential equation is the map,

$$(t,t_0,x_0)\mapsto \Phi(t,t_0,x_0)\,,$$

defined on a subset D of $\mathbb{R}\times\mathbb{R}\times M$, with values in M, such that, for each pair $(t_0,x_0)\in\Omega$, the map $t\mapsto\Phi(t,t_0,x_0)$ is the maximal solution of the differential equation which takes the value x_0 at $t = t_0$. Thus, by definition,

$$\frac{\partial \Phi(t,t_0,x_0)}{\partial t} = X\bigl(t, \Phi(t,t_0,x_0)\bigr)\,, \quad \Phi(t_0,t_0,x_0) = x_0\,.$$

We recall that a solution of a differential equation, defined on an open interval I, is said to be *maximal* if it is not the restriction of a solution defined on an open interval strictly larger than I.

It is well known that the map Φ is smooth; the subset D on which it is defined is open in $\mathbb{R} \times \mathbb{R} \times M$; for each $(t_0,x_0) \in \Omega$, the set of reals t such that (t,t_0,x_0) belongs to D is an open interval $I_{(t_0,x_0)}$ which contains t_0. In addition, for any $(t_0,x_0) \in \Omega$, t_1 and $t_2 \in \mathbb{R}$,

$$\Phi\bigl(t_2,t_1,\Phi(t_1,t_0,x_0)\bigr) = \Phi(t_2,t_0,x_0)\,.$$

More precisely, if the left-hand side of this equality is defined, *i.e.*, if $(t_1,t_0,x_0) \in D$ and $\bigl(t_2,t_1,\Phi(t_1,t_0,x_0)\bigr) \in D$, then the right-hand side is defined, *i.e.*, $(t_2,t_0,x_0) \in D$, and the equality holds. Conversely, if the right-hand side and $\Phi(t_1,t_0,x_0)$ are defined, *i.e.*, if both $(t_2,t_0,x_0) \in D$ and $(t_1,t_0,x_0) \in D$, then the left-hand side is defined, *i.e.*, $\bigl(t_2,t_1,\Phi(t_1,t_0,x_0)\bigr) \in D$, and the equality holds.

The proof of the existence of maximal solutions, and therefore the proof of the existence of the flow of a smooth differential equation, rests on the axiom of choice.

2.2 The manifold of motions of a smooth dynamical system

The concept of manifold of motions of a smooth dynamical system is closely related to the concept of flow. The *manifold of motions* of a smooth dynamical system is the set $\widetilde{M}$ of all the maximal solutions $t \mapsto \varphi(t)$ of the corresponding differential equation.

As a set, $\widetilde{M}$ is the quotient of the open subset Ω of $\mathbb{R} \times M$, on which the time-dependent vector field X is defined, by the equivalence relation,

(t_2,x_2) and $(t_1,x_1) \in \Omega$ are equivalent if (t_2,t_1,x_1) belongs to the open subset D of $\mathbb{R} \times \mathbb{R} \times M$ on which the flow Φ is defined and

$$x_2 = \Phi(t_2,t_1,x_1)\,.$$

A smooth manifold structure on $\widetilde{M}$ can be defined as follows. An element $a_0 \in \widetilde{M}$ is an equivalence class for the above-defined equivalence relation. Let $(t_0,x_0) \in \Omega$ be an element of this equivalence class. According to the theorem of global existence and uniqueness of the maximal solution of a smooth ordinary differential equation which satisfies given Cauchy data, there exists an open neighborhood $U_{(t_0,x_0)}$ of x_0 in M such that, for each $x \in U_{(t_0,,x_0)}$, there exists a unique maximal solution a which takes the value x at $t = t_0$. We can use the map $x \mapsto a$ to build a chart of $\widetilde{M}$, whose domain is diffeomorphic to $U_{(t_0,x_0)}$. Using this construction for all $(t_0,x_0) \in \Omega$, we obtain an atlas of $\widetilde{M}$, therefore a smooth manifold structure on this set, for which each point in $\widetilde{M}$ has an open neighborhood diffeomorphic to an open subset of M. The resulting smooth manifold structure of $\widetilde{M}$ need not be Hausdorff.

When the flow Φ is defined on $\mathbb{R} \times \mathbb{R} \times M$, the manifold of motions $\widetilde{M}$ is globally diffeomorphic to M. But there is no canonical diffeomorphism of $\widetilde{M}$ onto M: the choice of a particular time $t_0 \in \mathbb{R}$ determines a diffeomorphism of $\widetilde{M}$ onto M which associates with each motion $a \in \widetilde{M}$ the point $a(t_0) \in M$. Of course this diffeomorphism depends on t_0.

2.3 The modified flow

The value $\Phi(t,t_0,x_0)$ of the flow Φ at time t depends on t and on the *equivalence class* $a \in \widetilde{M}$ of $(t_0,x_0) \in \mathbb{R} \times M$, rather than on t_0 and x_0 separately. Indeed, if (t_0,x_0) and (t_1,x_1) are equivalent, $x_1 = \Phi(t_1,t_0,x_0)$ and $\Phi(t,t_1,x_1) = \Phi\bigl(t,t_1,\Phi(t_1,t_0,x_0)\bigr) = \Phi(t,t_0,x_0)$. The *modified flow* of the differential equation is the map, defined on an open subset of $\mathbb{R} \times \widetilde{M}$,

$$(t,a) \mapsto \widetilde{\Phi}(t,a) = \Phi(t,t_0,x_0)\,,$$

where $(t_0,x_0) \in \mathbb{R} \times M$ is any element of the equivalence class $a \in \widetilde{M}$. As explained below, Lagrange used this concept in a paper [16] published in 1809.

3 The motion of the planets in the solar system

During the eighteenth century, the quantitative description of the motion of the planets in the solar system was a major challenge for mathematicians and astronomers. Let us briefly indicate the state of the art on this subject before the publications of Lagrange and Poisson.

3.1 Kepler's laws

In 1609, using the measurements obtained by the Danish astronomer Tycho Brahe (1546–1601), the German astronomer and physicist Johannes Kepler (1571–1630) discovered two of the three laws which very accurately describe the motion of the planets in the solar system. His discovery of the third law of motion followed in 1611.

Kepler's first law

As a first (and very good) approximation, the orbit of each planet in the solar system is an ellipse, with the sun at one of its foci.

Kepler's second law

As a function of time, the motion of each planet is such that the area swept by the line segment which joins the planet to the sun increases linearly with time.

Kepler's third law

The ratio of the squares of the rotation periods of two planets in the solar system is equal to the ratio of the cubes of the major axes of their orbits.

3.2 Orbital elements of a planet

In Kepler's approximation, the motion of each planet in the solar system is a solution of a smooth dynamical system known as *Kepler's problem*, the motion of a massive point in a central attractive gravitational field, in Euclidean three-dimensional space. This problem, first formulated in mathematical terms by Isaac Newton (1642–1727), was solved by him in 1679 and published in 1687 in his *Philosophiae naturalis principia mathematica* [22]. Each possible motion is determined by six quantities, called the *orbital elements* of the planet. Altogether these six elements constitute a system of local coordinates on the manifold of motions of Kepler's problem, and therefore the dimension of this manifold is 6. Let us explain why.

First, we must determine the plane containing the planet's orbit. It is a plane containing the attractive center the sun. Such a plane may be determined, for example, by a unit vector with the sun as its origin, normal to this plane, *i.e.*, a point of the 2-dimensional unit sphere centered at the sun. For this, we need 2 orbital elements. The choice of such a vector simultaneously determines an *orientation* of the orbital plane, which will be assumed such that the planet rotates counter-clockwise around the sun.

We need two more orbital elements, which determine the orbit's shape and position in its plane. We know that this orbit is an ellipse with the sun as a focus. So the shape and position of the orbit are completely determined by the *excentricity vector*, discovered by Jakob Hermann (1678–1753), sometimes improperly called the *Laplace vector* or the *Lenz vector* (see A. Guichardet's paper [4]). It is the dimensionalless vector contained in the orbit's plane, directed from the attractive center towards the planet's perihelion, whose length is equal to the orbit's excentricity.

We still need an orbital element to determine the size of this elliptic orbit. For example, we may choose the length of its major axis.

Up to now we have seen that five orbital elements are needed to determine the planet's orbit. A sixth and final orbital element will determine the planet's position on its orbit. We may, for example, choose the point on this orbit at which the planet is at a fixed particular time. The second and third Kepler laws then fully determine the planet's position at all times, past, present and future.

3.3 The manifold of motions of a planet in Kepler's approximation

A modern and general description of the manifold of motions of a planet in Kepler's approximation was made by J.-M. Souriau [27]. He considered all possible motions, parabolic and hyperbolic as well as elliptic. By using a transform called *regularization of collisions*, he even included singular motions, in which the planet moves along a straight line until it collides, at a finite time, with the sun, and their analogues when time is inverted, in which a planet is, at a given time, ejected by the sun. This manifold is 6-dimensional. Due to the singular motions, it is non-Hausdorff. Other modern treatments of Kepler's dynamical system may be found in the books by V. Guillemin and S. Sternberg [5] and by B. Cordani [3].

Lagrange and Poisson were only interested in the elliptic motions of planets, not in parabolic or hyperbolic motions, which would be motions of comets rather than of planets. In Kepler's approximation, the set of all elliptic motions of a planet, excluding singular motions, is an open, connected, Hausdorff submanifold of the manifold of all motions. We have seen above that the six orbital elements of the planet constitute a system of local coordinates on this manifold. If we choose a particular value of the time and consider the three coordinates of the planet and the three components of its velocity at that time, in any space reference frame, we obtain another system of local coordinates on the manifold of motions, and another way of showing that its dimension is 6.

3.4 Beyond Kepler's approximation

Kepler's approximation is only valid under the assumptions that each planet interacts gravitationally exclusively with the sun, and that its mass is negligible compared with that of the sun. In fact, even if one does not take into account the gravitational interaction between planets, unless one assumes that the mass of the planet is negligible compared with that of the sun, its orbit is an ellipse whose focus is the *center of mass* of the system planet-sun, not the center of the sun. This center of mass is different for each planet. Therefore the planets have two kinds of gravitational interactions: their direct mutual interactions, and the interaction that each of them exerts on all the other planets through its interaction with the sun.

To go beyond Kepler's approximation, astronomers and mathematicians used a very natural idea: each planet was considered to be moving around the sun on an ellipse whose orbital elements slowly vary in time, instead of remaining rigorously constant, as in Kepler's approximation. While astronomers increased the accuracy of their observations, from which they deduced tables for these slow variations, mathematicians sought to calculate them, using Newton's law of gravitational interaction. Let us briefly indicate some important stages of this search, which finally led to the mathematical discoveries made by Lagrange and Poisson.

In 1773, Pierre-Simon Laplace (1749–1827) proved that, up to the first order, the gravitational interactions between the planets cannot produce secular variations in their periods, nor in the length of their orbit's major axis [19]. Then in 1774, after reading Lagrange's paper discussed below, he calculated the slow variations of other orbital elements, the excentricity and the aphelion's position [20]. In [21] he improved on his results for the planets Jupiter and Saturn.

In 1774, Lagrange calculated the variations of the position of the nodes and orbital inclinations of the planets [11]. Then, in several papers presented to the Berlin Academy of Sciences between 1776 and 1784, he improved on Laplace's results, and determined the slow variations of other orbital elements. He distinguished between *secular variations*, non-periodic, or periodic with very long periods, which may become large with time [12, 13], and *periodic variations*, which remain bounded for all times [14]. He proved, with fewer approximations than Laplace, that the gravitational interactions between the planets cannot produce secular variations of their periods.

Then, it seems that for more than 20 years, Lagrange ceased being interested in the subject, and he published no important new results on the motion of the planets.

On 20 June 1808, Poisson presented a paper, *Sur les inégalités séculaires des moyens mouvements des planètes*, to the French Academy of Sciences [25], in which he removed a simplifying assumption that had been made by Lagrange in his papers of the years 1776–1784 on the variation of the periods of the planets.

Stimulated by Poisson's contribution, Lagrange returned to the problem in his *Mémoire sur la théorie des variations des éléments des planètes*, presented to the French Academy of Sciences on 22 August 1808 [15]. We quote a passage from his introduction which shows that he clearly understood that Poisson's result was due to a still hidden mathematical structure. After recalling Laplace's important result of 1773, and the improvements he had obtained in 1776, he wrote,

On n'avait pas été plus loin sur ce point; mais M. Poisson y a fait un pas de plus dans le Mémoire qu'il a lu il y a deux mois à la Classe, sur les inégalités séculaires des moyens mouvements des planètes, et dont nous avons fait le rapport dans la dernière séance. Il a poussé l'approximation de la même formule jusqu'aux termes affectés des carrés et des produits des masses, en ayant égard dans cette formule à la variation des éléments que j'avais regardés comme constants dans la première approximation.... il parvient d'une manière ingénieuse à faire voir que ces sortes de termes ne peuvent non plus produire dans le grand axe de variations proportionnelles au temps. ... Il me parut que le résultat qu'il venait de trouver par le moyen des formules qui représentent le mouvement elliptique était un résultat analytique dépendant de la forme des équations différentielles et des conditions de la variabilité des constantes, et qu'on devait y arriver par la seule force de l'Analyse, sans connaître les expressions particulières des quantités relatives à l'orbite elliptique.

On 13 March 1809, Lagrange extended his method in the *Mémoire sur la théorie générale de la variation des constantes arbitraires dans tous les problèmes de mécanique* [16].

On 16 October 1809, Poisson presented his paper, *Sur la variation des constantes arbitraires dans les questions de mécanique* [26]. It is in this work, which is devoted to the subject that had been considered by Lagrange just a few months earlier, that he solved a question which had been left in abeyance by Lagrange, and introduced the composition law today called the *Poisson bracket*.

On 19 February 1810, Lagrange presented his *Second mémoire sur la théorie de la variation des constantes arbitraires dans les problèmes de mécanique* [17]. Here, he recognizes Poisson's contribution, but claims that the main ideas were already contained in his own previous paper. Lagrange included a simplified presentation of the method of varying constants in the second edition, published in 1811, of his *Mécanique analytique* [18].

It appears that, more than twenty years later, these works were still not fully understood, since Augustin Louis Cauchy (1789–1857) had to give a very clear presentation of Lagrange's method in his *Note sur la variation des constantes arbitraires dans les problèmes de mécanique* [2]. Published in 1837, this paper is a summary of a longer work he had presented in 1831 to the Academy of Sciences of Turin. In this work, Cauchy makes essential use of the formalism recently introduced by Sir William Rowan Hamilton (1805–1865) in his papers *On a general method in Dynamics* [6] and *Second essay on a general method in Dynamics* [7].

3.5 Remarks on the stability of the solar system

At the beginning of the nineteenth century, the results obtained by Laplace, Lagrange and Poisson concerning the absence of secular variations of the major axis of orbits of the planets were regarded as a proof of the stability of the solar system. Today we know that these results were not rigorously established [1]. In his famous article [23], Jules Henri Poincaré (1854–1912) proved that the problem is much more subtle than Laplace, Lagrange and Poisson had imagined. In [24], vol. III, chapter XXVI, page 140, he remarks that Poisson's results do not exclude the existence of terms of the form $At\sin(\alpha t+\beta)$ in the expression for the variations of the major axis of a planet's orbit, t being the time, A, α and β denoting constants. Such terms can take very large values for large values of the time, although they vanish periodically. In fact, the problem of the stability of the solar system gave rise to modern research by many mathematicians, notably Andrei Nicolaievich Kolmogorov (1909–1987), Jürgen Moser (1928–1999), Vladimir Igorevich Arnol'd (1937), and Nikolai Nikolaievich Nekhoroshev (1946–2008).

4 The method of varying constants

4.1 Lagrange's paper of 1809

Lagrange considers a mechanical system with kinetic energy,

$$T = T(r,s,u,\ldots,r',s',u'\ldots)\,,$$

where r, s, u, … are independent real variables which describe the system's position in space. For a planet moving around the sun, these variables are the three coordinates of the planet in some reference frame. Let n be the number of these variables. In modern terms, n is the dimension of the *configuration manifold*. The quantities r', s', u', …, are the derivatives of r, s, u, …, with respect to the time, t,

$$r' = \frac{dr}{dt}, \quad s' = \frac{ds}{dt}, \quad u' = \frac{du}{dt}, \quad ,\ldots$$

As a first approximation, Lagrange assumes that the forces which act on the system derive from a potential V, which depends on r, s, u, …, but not on the time derivatives, r', s', u', … For a planet's motion, V is the gravitational potential due to the sun's attraction. The equations which determine the motion, established by Lagrange in his *Mécanique analytique* [18], are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = 0\,,$$

and similar equations in which r and r' are replaced by s and s', u and u', …

The general solution of this system of n second-order equations depends on the time t and on $2n$ integration constants. Lagrange denotes these constants by a, b, c, f, g, h,…, and writes this general solution as

$$r = r(t,a,b,c,f,g,h,\ldots), \quad s = s(t,a,b,c,f,g,h,\ldots), \quad u = \cdots\,.$$

For a planet's motion, the $2n$ integration constants a, b, c, f, g, h, $\dots$ are the *orbital elements* of the planet.

At a second approximation, Lagrange assumes that the potential V does not fully describe the forces which act on the system, and should be replaced by $V - \Omega$, where Ω may depend on r, s, u, $\dots$, and on the time t. For a planet's motion, Ω describes the gravitational interactions between the planet under consideration and all the other planets, which had been considered to be negligible in the first approximation. Ω depends on the time, because the planets which are the source of these gravitational interactions are in motion. The equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial r'}\right) - \frac{\partial T}{\partial r} + \frac{\partial V}{\partial r} = \frac{\partial \Omega}{\partial r},$$

and similar equations in which r and r' are replaced by s and s', u and u', $\dots$

Lagrange writes the solution of this new system under the form

$$r = r\Big(t, a(t), b(t), c(t), f(t), g(t), h(t), \dots\Big),$$

and similar expressions for s, u, $\dots$. The function

$$(t, a, b, c, f, g, h, \dots) \mapsto r(t, a, b, c, f, g, h \dots)$$

which appears in this expression, and the analogous functions which appear in the expressions of s, u, $\dots$ are, of course, those previously found when solving the problem in its first approximation, with Ω set to 0. These functions are therefore considered as *known*.

It only remains to find the $2n$ functions of the time $t \mapsto a(t)$, $t \mapsto b(t)$, $\dots$. These functions will depend on the time and on $2n$ arbitrary integration constants.

4.2 Lagrange parentheses

Lagrange obtains the differential equations which determine the time variations of these functions $a(t)$, $b(t)$, $\dots$. The calculations by which he obtains these equations are at first very complicated, and he makes two successive improvements, first in an *Addition*, then in a *Supplément* to his initial paper. He finds a remarkable property: these equations become very simple when they are expressed in terms of quantities that he denotes by (a,b), (a,c), (a,f), (b,c), (b,f), $\dots$. Today, these quantities are still in use and they are called *Lagrange parentheses*.

Lagrange parentheses are functions of a, b, c, f, g, h, $\dots$. They do not depend on time, nor on the additional forces which act on the system when Ω is taken into account. Jean-Marie Souriau [27, 28] has shown that they are the components of the *canonical symplectic* 2*-form* on the manifold of motions of the mechanical system, in the chart of this manifold whose local coordinates are a, b, c, f, g, h, $\dots$. So Lagrange discovered the notion of a symplectic structure more than 100 years before that notion was so named by Hermann Weyl [29].

We stress the fact that the Lagrange parentheses are relative to the mechanical system with kinetic energy T and applied forces described by the potential V. The additional forces described by Ω play no part in Lagrange's parentheses: the consideration of these additional forces permitted the discovery of a structure in which they play no part!

At first, Lagrange obtained very complicated expressions for the parentheses (a,b), (a,c), (b,c), ... In the *Addition* to his paper (Section 26 of [17]), he obtained the much simpler expressions:

$$(a,b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots\,,$$

and similar expressions for (a,c), (b,c), ... We have used the notations introduced by Hamilton [6, 7] and Cauchy [2] thirty years later,

$$p_r = \frac{\partial T}{\partial r'}\,,\quad p_s = \frac{\partial T}{\partial s'}\,,\quad p_u = \frac{\partial T}{\partial u'}\,,$$

while Lagrange used the less convenient notations, T', T'' and T''', instead of p_r, p_s and p_u.

We recall that r, s, u, ... are local coordinates on the configuration manifold of the system, and r', s', u' their partial derivatives with respect to time. The kinetic energy T, which depends on r, s, u, ..., r', s', u', ..., is a function defined on the tangent bundle of the configuration manifold, which is called the *manifold of kinematic states* of the system.

The map

$$(r,s,u,\ldots,r',s',u',\ldots) \mapsto (r,s,u,\ldots,p_r,p_s,p_u,\ldots)\,,$$

called the *Legendre transformation,* is defined on the tangent bundle of the configuration manifold, and takes values in the cotangent bundle of this manifold, called the *phase space* of the system. When the kinetic energy is a positive definite quadratic form, this map is a diffeomorphism. This occurs very often, for example in the motion of a planet around the sun, the mechanical system considered by Lagrange.

Since the integration constants a, b, c, f, g, h, ... constitute a system of local coordinates on the manifold of motions, they completely determine the motion of the system. We again stress that this system is a *first approximation*, where Ω is set to 0. Therefore, for each time t, the instantaneous values of the quantities r, s, u, ..., r', s', u', ..., are determined as soon as a, b, c, f, g, h, ... are given.

Conversely, the existence and uniqueness theorem for solutions of ordinary differential equations (implicitly considered as obvious by Lagrange, at least for Kepler's problem whose solutions are explicitly known) shows that when the values of r, s, u, ..., r', s', u', ... at any given time t are known, then the motion is determined, so a, b, c, f, g, h, ... are known.

In short, for each time t, the map which associates to a motion of coordinates $(a,\ b,\ c,\ f,\ g,\ h,\ldots)$ the values at time t of $(r,\ s,\ u,\ldots,r',\ s',\ u',\ldots)$ is a diffeomorphism from the manifold of motions onto the manifold of kinematic states of the system. The composition of this diffeomorphism with the Legendre transformation yields, for each time t, a diffeomorphism from the manifold of motions onto the phase space,

$$(a,\ b,\ c,\ f,\ g,\ h,\dots) \mapsto \bigl(r(t),\ s(t),\ u(t),\dots,\ p_r(t),\ p_s(t),\ p_u(t),\dots\bigr)\,,$$

where $r(t)$, $s(t)$, $u(t)$, $p_r(t)$, $p_s(t)$, $p_u(t)$ are the values taken at time t by the corresponding quantities.

The partial derivatives in the expression of the Lagrange parentheses are the partial derivatives of the diffeomorphism

$$(a,\ b,\ c,\ f,\ g,\ h,\dots) \mapsto (r(t),\ s(t),\ u(t),\dots,\ p_r(t),\ p_s(t),\ p_u(t),\dots)$$

where t is any value of the time, considered as fixed.

Important remark

The Lagrange parentheses (a,b) are defined when a complete system of local coordinates $(a,\ b,\ c,\ f,\ g,\ h,\dots)$ has been chosen on the manifold of motions: the value of (a,b) depends not only on the values of the functions a and b on that manifold, but also on all the other coordinate functions, $c,\ f,\ g,\ h,\ \dots$.

4.3 The canonical symplectic form

Let us again consider the diffeomorphism

$$(a,\ b,\ c,\ f,\ g,\ h,\dots) \mapsto (r(t),\ s(t),\ u(t),\dots,\ p_r(t),\ p_s(t),\ p_u(t),\dots)\,,$$

where t is any value of the time, considered as fixed. The exterior calculus of differential forms, created by Élie Cartan at the beginning of the twentieth century, did not exist in Lagrange's times. Today, with this very efficient tool, it is easy to prove that the Lagrange parentheses are the components of the pull-back by this diffeomorphism, on the manifold of motions, of the canonical symplectic 2-form of the cotangent bundle of the configuration manifold. In fact,

$$\begin{aligned}
(a,b)\,da\wedge db &+ (a,c)\,da\wedge dc+\cdots+(b,c)\,db\wedge dc+\cdots\\
&= \left(\frac{\partial r}{\partial a}da+\frac{\partial r}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_r}{\partial a}da+\frac{\partial p_r}{\partial b}db+\cdots\right)\\
&\quad+\left(\frac{\partial s}{\partial a}da+\frac{\partial s}{\partial b}db+\cdots\right)\wedge\left(\frac{\partial p_s}{\partial a}da+\frac{\partial p_s}{\partial b}db+\cdots\right)+\dots\\
&= dr\wedge dp_r+ds\wedge dp_s+du\wedge dp_u+\cdots\,.
\end{aligned}$$

The last expression is the well known formula for the components of a symplectic 2-form in Darboux coordinates.

Another important remark

Lagrange proved that, although they are defined by means of a diffeomorphism which depends on time, the parentheses he introduced *do not depend explicitly on time*: they are functions on the manifold of motions. When proving this result, Lagrange proved that *the canonical symplectic* 2*-form on phase space is invariant under the flow of the evolution vector field on this space*.

4.4 Formulae for the variation of constants

Lagrange proved that the derivatives with respect to time of the "constants that are varied" , *a*, *b*, ..., satisfy

$$\sum_{j=1}^{2n}(a_i,a_j)\frac{da_j}{dt}=\frac{\partial\Omega}{\partial a_i}\,,\quad 1\leq i\leq 2n\,,$$

where, for short, I have written a_i, $1\leq i\leq 2n$, instead of *a*, *b*, *c*, ..., and where I have taken into account the skew-symmetry, $(a_j,a_i)=-(a_i,a_j)$.

Lagrange indicates that by solving this linear system, one obtains something like

$$\frac{da_i}{dt}=\sum_{j=1}^{2n}L_{ij}\frac{\partial\Omega}{\partial a_j}\,,\quad 1\leq i\leq 2n\,.$$

He explains that the L_{ij} are functions of the a_i which do not depend explicitly on time. In modern terms, the L_{ij} are functions defined on the manifold of motions. But Lagrange does not state their explicit expressions. That would be done by Poisson a few months later.

4.5 Poisson's paper of 1809 and the Poisson bracket

When he was a student at the École Polytechnique, Poisson attended lectures by Lagrange. In a paper read before the French Academy of Sciences on 16 October 1809 [26], he added an important ingredient to Lagrange's *method of varying constants*. He introduced new quantities, defined on the manifold of motions, which he denoted by (a,b), (a,c), ... These quantities *are not* the Lagrange parentheses. Today, they are called the *Poisson brackets*. In his paper, Poisson also uses Lagrange parentheses but he denotes them differently, by $[a,b]$ instead of (a,b), $[a,c]$ instead of (a,c), etc.

We shall retain Lagrange's notations (a,b), (a,c), ... for the Lagrange parentheses and we will denote the Poisson brackets by $\{a,b\}$, $\{a,c\}$, ...

The expression of the Poisson brackets is

$$\{a,b\}=\frac{\partial a}{\partial p_r}\frac{\partial b}{\partial r}-\frac{\partial a}{\partial r}\frac{\partial b}{\partial p_r}+\frac{\partial a}{\partial p_s}\frac{\partial b}{\partial s}-\frac{\partial a}{\partial s}\frac{\partial b}{\partial p_s}+\frac{\partial a}{\partial p_u}\frac{\partial b}{\partial u}-\frac{\partial a}{\partial u}\frac{\partial b}{\partial p_u}+\cdots\,.$$

Of course $\{a,c\}$, $\{b,c\}$, ... are given by similar formulae. We observe that in these formulae there appear the partial derivatives of the local coordinates *a*, *b*, *c*, ... on the manifold of motions, considered as functions of the dynamical state of the system at a fixed time, t. The independent variables which describe this dynamical state are the values, at time t, of the quantities r, p_r, s, p_s, u, p_u, ...

The above formula is the well known expression of the Poisson bracket of two functions a and b defined on a symplectic manifold, in Darboux coordinates.

4.6 Poisson brackets versus Lagrange parentheses

Let us compare the Poisson bracket

$$\{a,b\} = \frac{\partial a}{\partial p_r}\frac{\partial b}{\partial r} - \frac{\partial a}{\partial r}\frac{\partial b}{\partial p_r} + \frac{\partial a}{\partial p_s}\frac{\partial b}{\partial s} - \frac{\partial a}{\partial s}\frac{\partial b}{\partial p_s} + \frac{\partial a}{\partial p_u}\frac{\partial b}{\partial u} - \frac{\partial a}{\partial u}\frac{\partial b}{\partial p_u} + \cdots$$

with the Lagrange parenthesis

$$(a,b) = \frac{\partial r}{\partial a}\frac{\partial p_r}{\partial b} - \frac{\partial r}{\partial b}\frac{\partial p_r}{\partial a} + \frac{\partial s}{\partial a}\frac{\partial p_s}{\partial b} - \frac{\partial s}{\partial b}\frac{\partial p_s}{\partial a} + \frac{\partial u}{\partial a}\frac{\partial p_u}{\partial b} - \frac{\partial u}{\partial b}\frac{\partial p_u}{\partial a} + \cdots .$$

We see that these formulae involve the partial derivatives of two diffeomorphisms which are inverses of one another: the Poisson bracket involves the partial derivatives of the coordinates a, b, ... on the manifold of motions with respect to the coordinates r, s, r', s', ... on the phase space, while the Lagrange parenthesis involves the partial derivatives of r, s, r', s', ... with respect to a, b, ...

We have seen that Lagrange's parentheses (a,b), (a,c), ..., are the components of the symplectic 2-form on the manifold of motions, in the chart of this manifold whose local coordinates are a, b, c, ... The Poisson brackets $\{a,b\}$, $\{a,c\}$, ..., are the components in the same chart of the associated Poisson bivector.

The matrix whose components are the Lagrange parentheses (a,b), (a,c), ..., and the matrix whose components are the Poisson brackets $\{a,b\}$, $\{a,c\}$, ..., are *inverses of one another*. This property was clearly stated by Cauchy in his paper [2], read before the Academy of Turin on 11 October 1831, 22 years after the publication of Lagrange's and Poisson's papers.

4.7 Remark about the Poisson theorem

The Poisson theorem states that the Poisson bracket of two first integrals, *i.e.*, functions which remain constant on each trajectory of the system, is a first integral. Today, this result is often presented as a consequence of the *Jacobi identity*. This identity was not known to Lagrange, nor to Poisson, who considered the constancy of the Poisson bracket of two first integrals to be due to the fact that it is a function defined on the manifold of motions. The Poisson bracket can indeed be defined for any pair of smooth functions on the manifold of motions, and it is still a function defined on that manifold which depends only on the two given functions. As stated above, contrary to the Poisson bracket, the Lagrange parenthesis can be defined only for a pair of functions which are part of a complete system of coordinate functions, and not for a pair of smooth functions in general.

4.8 The Jacobi identity

Lagrange and Poisson observed the skew-symmetry of their parentheses and brackets, but said nothing about the Jacobi identity for the Poisson brackets, nor about the relations between Lagrange's parentheses expressing the fact that they are the components of a *closed* 2-form.

Discovered by the German mathematician Carl Gustav Jacob Jacobi (1804–1851) [8, 10], the identity that bears his name involves three arbirary smooth functions f, g and h defined on a symplectic (or a Poisson) manifold,

$$\big\{f,\{g,h\}\big\}+\big\{g,\{h,f\}\big\}+\big\{h,\{f,g\}\big\}=0\,,$$

or three smooth vector fields X, Y and Z defined on a smooth manifold,

$$\big[X,[Y,Z]\big]+\big[Y,[Z,X]\big]+\big[Z,[X,Y]\big]=0\,.$$

Jacobi understood the importance of this identity, which later played an important part in the theory of Lie groups and Lie algebras developed by the Norwegian mathematician Marius Sophus Lie (1842–1899).

4.9 Lagrange's paper of 1810

In his paper [17], Lagrange gives simpler expressions of his previous results, using Poisson brackets. He writes the differential equations which determine the time variations of the "constants" $a, b, \ldots$, in the form

$$\frac{da_i}{dt}=\sum_{j=1}^{2n}\{a_i,a_j\}\frac{\partial\Omega}{\partial a_j}\,,\quad 1\leq i\leq 2n\,.$$

Here I have denoted the constants by a_i, $1\leq i\leq 2n$, a notation that allows the use of the symbol $\sum_{i=1}^{2n}$ for a more concise expression. Lagrange used longer expressions in which the constants were denoted by a, b, c, f, g, h, ...

Let us observe that Lagrange could have written his equations in the simpler form,

$$\frac{da_i}{dt}=\{a_i,\Omega\}\,,\quad 1\leq i\leq 2n\,,$$

since Ω can be considered as a function defined on the product of the manifold of motions with the factor $\mathbb{R}$, for the time. Therefore the Poisson bracket $\{a_i,\Omega\}$ can be unambiguously defined as

$$\{a_i,\Omega\}=\sum_{j=1}^{2n}\{a_i,a_j\}\frac{\partial\Omega}{\partial a_j}\,.$$

Lagrange did not use this simpler expression, nor did Poisson in his paper of 1809. Both used the Poisson bracket only for coordinate functions a_i, not for more general functions such as Ω.

4.10 Cauchy's paper of 1837

This short paper of 6 pages, published in the *Journal de Mathématiques pures et appliquées*, is extracted from the longer paper presented by Cauchy before the Academy of Turin on 11 October 1831. Its title is almost the same as those of the papers by Lagrange and Poisson.

Cauchy very clearly explains the main results due to Lagrange and Poisson, using Hamilton's formalism. However, he does not write Poisson brackets with the function Ω (which is denoted by R in his paper).

Cauchy proves, without using the word *matrix*, that the matrix whose coefficients are Lagrange's parentheses of some coordinate functions and the matrix whose coefficients are the Poisson brackets of the same coordinate functions are inverses of one another.

5 Varying constants revisited

I shall now present, in modern language and with today's notations, the main results due to Lagrange and Poisson concerning the method of varying constants. I will follow Cauchy's paper of 1837.

Let (M,ω) be a $2n$-dimensional symplectic manifold, with a Hamiltonian function $Q : M\times\mathbb{R}\to\mathbb{R}$ which may be time-dependent (Q is the notation used by Cauchy). Let M_0 be the manifold of motions of this Hamiltonian system, and let $\widetilde{\Phi} : \mathbb{R}\times M_0\to M$, $(t,a)\mapsto\widetilde{\Phi}(t,a)$ be the modified flow of the Hamitonian vector field associated with Q (in the sense of Section 2.3). The easiest way of writing Hamilton's equation is the following. For each smooth function, $g : M\to\mathbb{R}$,

$$\frac{\partial\bigl(g\circ\widetilde{\Phi}(t,a)\bigr)}{\partial t}=\{Q,g\}\bigl(\widetilde{\Phi}(t,a)\bigr)\,.$$

We now assume that the system's *true* Hamiltonian is $Q+R$ instead of Q, where R may be time-dependent.

The aim of the *method of varying constants* is to transform the modified flow of the Hamiltonian vector field associated with Q into the modified flow of the Hamiltonian vector field associated with $Q+R$. More precisely, the aim is to find a map $\Psi:\mathbb{R}\times M_1\to M_0$, $(t,b)\mapsto a=\Psi(t,b)$, where M_1 is the manifold of motions of the system with Hamiltonian $Q+R$, such that $(t,b)\mapsto\widetilde{\Phi}\bigl(t,\Psi(t,b)\bigr)$ is the modified flow of the vector field with Hamiltonian $Q+R$.

These maps must satisfy, for any smooth function $g : M\to\mathbb{R}$,

$$\frac{d}{dt}\Bigl(g\circ\widetilde{\Phi}\bigl(t,\Psi(t,b)\bigr)\Bigr)=\{Q+R,g\}\Bigl(\widetilde{\Phi}\bigl(t,\Psi(t,b)\bigr)\Bigr)\,.$$

For each value t_0 of the time t,

$$\frac{d}{dt}\Bigl(g\circ\widetilde{\Phi}\bigl(t,\Psi(t,b)\bigr)\Bigr)\Bigm|_{t=t_0}=\frac{d}{dt}\Bigl(g\circ\widetilde{\Phi}\bigl(t,\Psi(t_0,b)\bigr)\Bigr)\Bigm|_{t=t_0}+\frac{d}{dt}\Bigl(g\circ\widetilde{\Phi}\bigl(t_0,\Psi(t,b)\bigr)\Bigr)\Bigm|_{t=t_0}\,.$$

When t_0 is fixed, $\bigl(t,\Psi(t_0,b)\bigr) \mapsto \widetilde\Phi\bigl(t,\Psi(t_0,b)\bigr)$ is the flow of the vector field with Hamiltonian Q. Thus the first term of the right-hand side is

$$\frac{d}{dt}\Bigl(g\circ\widetilde\Phi\bigl(t,\Psi(t_0,b)\bigr)\Bigr)\Bigm|_{t=t_0} = \{Q,g\}\Bigl(\widetilde\Phi\bigl(t_0,\Psi(t_0,b)\bigr)\Bigr)\,.$$

Therefore the second term of the right-hand side must be

$$\begin{aligned}\frac{d}{dt}\Bigl(g\circ\widetilde\Phi\bigl(t_0,\Psi(t,b)\bigr)\Bigr)\Bigm|_{t=t_0} &= \Bigl(\{Q+R,g\}-\{Q,g\}\Bigr)\Bigl(\widetilde\Phi\bigl(t_0,\Psi(t_0,b)\bigr)\Bigr)\\ &= \{R,g\}_M\Bigl(\widetilde\Phi\bigl(t_0,\Psi(t_0,b)\bigr)\Bigr)\\ &= \{R\circ\widetilde\Phi_{t_0}, g\circ\widetilde\Phi_{t_0}\}_{M_0}\bigl(\Psi(t_0,b)\bigr)\,.\end{aligned}$$

The Poisson bracket of functions on M is denoted by $\{\ ,\ \}$ when there is no risk of confusion, and by $\{\ ,\ \}_M$ when we want to distinguish it from the Poisson bracket of functions defined on M_0, which is denoted by $\{\ ,\ \}_{M_O}$. For the last equality, we have used the fact that $\widetilde\Phi_{t_0} : M_0 \to M$ is a Poisson map.

The function $g_0 = g\circ\widetilde\Phi_{t_0}$ can be any smooth function on M_0, so the last equality may be written as

$$\Bigl\langle dg_0, \frac{\partial\Psi(t,b)}{\partial t}\Bigr\rangle\Bigm|_{t=t_0} = \frac{d\Bigl(g_0\bigl(\Psi(t,b)\bigr)\Bigr)}{dt}\Bigm|_{t=t_0} = \{R\circ\widetilde\Phi_{t_0}, g_0\}_{M_0}\bigl(\Psi(t_0,b)\bigr)\,.$$

Now, t_0 may take any value, so the last equation proves that for each $b\in M_1$, the manifold of motions of the system with Hamiltonian $Q+R$, the map $t\mapsto\Psi(t,b)$ is an integal curve, lying in the manifold M_0 of motions of the system with Hamiltonian Q, of the Hamiltonian system with the time-dependent Hamiltonian,

$$(t,a)\mapsto R\bigl(t,\widetilde\Phi(t,a)\bigr)\,,\quad (t,a)\in\mathbb{R}\times M_0\,.$$

This is the result discovered by Lagrange *circa* 1808. It is an *exact* result, not an approximate one. However, when the method is used for the determination of the motion of a given planet of the solar system, the potential R depends upon the positions of all the other planets, which are not known exactly. Therefore the method must be used in conjunction with successive approximations, the value of R used at each step being that deduced from the calculations made at the preceding steps.

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