Master symmetries
and bi-Hamiltonian structures
for the relativistic Toda lattice

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Abstract

We define a bi-Hamiltonian formulation for the relativistic Toda lattice with a recursion operator on $\mathbb{R}^{2n}$. We use a theorem of W. Oevel to generate higher order Poisson tensors and master symmetries for the relativistic Toda lattice. These Poisson tensors and master symmetries reduce to $\mathbb{R}^{2n-1}$.

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Introduction

The relativistic Toda lattice (RTL) was introduced by S. N. Ruijsenaars [1] and has been studied by many authors, in particular M. Bruschi and O. Ragnisco [2, 3], W. Oevel et al. [4], Yu. B. Suris [5] and P. Damianou [6]. It is a finite-dimensional completely integrable bi-Hamiltonian system. Its bi-Hamiltonian formulation and its complete integrability were proven by using various methods: Lax representation [3], [6], master symmetries [4], [6], recursion operators [2], [4].

Master symmetries were introduced by A. S. Fokas and B. Fuchssteiner [9] and were also studied by W. Oevel and B. Fuchssteiner [10] and B. Fuchssteiner [11].

In this paper we obtain a bi-Hamiltonian formulation for the RTL by introducing two compatible Poisson tensors on $\mathbb{R}^{2n}$ which, by a suitable projection map onto $\mathbb{R}^{2n-1}$, reduce to the two compatible Poisson tensors of the RTL. Since one of the Poisson structures introduced on $\mathbb{R}^{2n}$ is nondegenerate, we have a recursion operator and the bi-Hamiltonian structure of the RTL is in fact multi-Hamiltonian. Then, using a method introduced by R. L. Fernandes [7] for the non-relativistic Toda lattice, based on a theorem due to W. Oevel [8], we determine master symmetries for the RTL. Our results answer a question put by P. Damianou in [6].

In this paper, all the manifolds, maps, vector and tensor fields are assumed to be smooth. Let us recall that a bi-Hamiltonian manifold is a manifold $M$ equipped with two compatible Poisson tensors $\Lambda_0$ and $\Lambda_1$; it is denoted by $(M, \Lambda_0, \Lambda_1)$. A vector field $X$ on $M$ is said to be a bi-Hamiltonian vector field if it is Hamiltonian with respect to both Poisson structures. A recursion operator for $(M, \Lambda_0, \Lambda_1)$ is a vector bundle map $R : TM \to TM$ such that $\Lambda^2_1 = R \circ \Lambda^2_0$, where $\Lambda^0_0 : T^*M \to TM$ and $\Lambda^2_1 : T^*M \to TM$ are the vector bundle maps associated with the Poisson tensors $\Lambda_0$ and $\Lambda_1$. A bi-Hamiltonian manifold for which there exists a recursion operator is called a Poisson-Nijenhuis manifold [12].

1 A bi-Hamiltonian formulation for the relativistic Toda lattice

We consider $\mathbb{R}^{2n}$ with coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ and the canonical Poisson tensor

$$\Lambda_1 = \sum_{i=1}^{n} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}. \quad (1)$$

Following Yu. B. Suris [5], we take $(c_1, \ldots, c_{n-1}, d_1, \ldots, d_n)$ as variables on $\mathbb{R}^{2n-1}$, with

$$c_i = \exp(q^i - q^{i+1} + p_i) \quad \text{and} \quad d_i = \exp(p_i) \quad (2)$$

and we denote by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ the map

$$\pi : (q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n). \quad (3)$$
The relativistic Toda lattice is a finite-dimensional integrable system. The equations of motion of the RTL are

\[
\begin{align*}
\dot{c}_i &= c_i(d_{i+1} - d_i + c_{i+1} - c_{i-1}) \\
\dot{d}_i &= d_i(c_i - c_{i-1}),
\end{align*}
\]

where \(i = 1, \ldots, n\) and, by convention, \(c_0 = c_n = 0\).

The RTL is a bi-Hamiltonian system with respect to the following compatible Poisson tensors on \(\mathbb{R}^{2n-1}\),

\[
\tilde{\Lambda}_0 = \sum_{i=1}^{n-1} c_i \left( \frac{\partial}{\partial c_i} \wedge \left( \frac{\partial}{\partial d_i} - \frac{\partial}{\partial d_{i+1}} \right) + \frac{\partial}{\partial d_i} \wedge \frac{\partial}{\partial d_{i+1}} \right)
\]

and

\[
\tilde{\Lambda}_1 = \sum_{i=1}^{n-1} c_i \frac{\partial}{\partial c_i} \wedge \left( -c_{i+1} \frac{\partial}{\partial c_{i+1}} + d_i \frac{\partial}{\partial d_i} - d_{i+1} \frac{\partial}{\partial d_{i+1}} \right),
\]

and the bi-Hamiltonian vector field

\[
\tilde{\Lambda}_0^\#(d\tilde{H}_1) = \tilde{\Lambda}_1^\#(d\tilde{H}_0)
\]

\[
\tilde{H}_0 = \sum_{i=1}^{n} (c_i + d_i) \quad \text{and} \quad \tilde{H}_1 = \sum_{i=1}^{n} (c_{i-1}(c_i + d_i) + \frac{1}{2}(c_i + d_i)^2),
\]

where, by convention, \(c_0 = c_n = 0\) in (6), (7) and (8).

We remark that the Poisson bracket associated with \(\tilde{\Lambda}_0\) is linear:

\[
\{c_i, d_i\}_0 = c_i; \quad \{c_i, d_{i+1}\}_0 = -c_i; \quad \{d_i, d_{i+1}\}_0 = c_i,
\]

while the Poisson bracket corresponding to \(\tilde{\Lambda}_1\) is quadratic:

\[
\{c_i, c_{i+1}\}_1 = -c_i; \quad \{c_i, d_i\}_1 = c_i; \quad \{c_i, d_{i+1}\}_1 = -c_i d_{i+1}.
\]

A simple computation shows that the map

\[
\pi : (\mathbb{R}^{2n}, \Lambda_0) \to (\mathbb{R}^{2n-1}, \tilde{\Lambda}_1)
\]

is a Poisson morphism, that is, the canonical Poisson tensor \(\Lambda_0\) on \(\mathbb{R}^{2n}\) reduces to the quadratic Poisson tensor \(\tilde{\Lambda}_1\) on \(\mathbb{R}^{2n-1}\). This is not the case of the non-relativistic Toda lattice where the canonical Poisson bracket associated with \(\Lambda_1\) reduces to a Poisson bracket on \(\mathbb{R}^{2n-1}\), that is linear.
Our goal is to provide a bi-Hamiltonian formulation on $\mathbb{R}^{2n}$ for the RTL with a recursion operator.

The first step is to define a Poisson tensor on $\mathbb{R}^{2n}$, compatible with the canonical Poisson tensor $\Lambda_1$ and that reduces to the linear Poisson tensor $\tilde{\Lambda}_0$ on $\mathbb{R}^{2n-1}$ given by (5).

**Proposition 1.1** Let $\Lambda_0$ be the following bivector on $\mathbb{R}^{2n}$:

$$
\Lambda_0 = \sum_{i=1}^{n} \exp(-p_i) \frac{\partial}{\partial q^i} \wedge \left( \frac{\partial}{\partial p_i} + \sum_{j=i+1}^{n} \frac{\partial}{\partial q^j} \right) + \sum_{i=1}^{n-1} \exp(q^i - q^{i+1} - p_{i+1}) \left( \left( \frac{\partial}{\partial p_i} + \frac{\partial}{\partial q^{i+1}} \right) \wedge \left( \frac{\partial}{\partial p_{i+1}} + \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j} \right) \right) - \frac{\partial}{\partial p_{i+1}} \wedge \sum_{j=i+2}^{n} \frac{\partial}{\partial q^j} \right).$$

(9)

Then,

(i) $[\Lambda_0, \Lambda_0] = 0$ and $[\Lambda_0, \Lambda_1] = 0$;

(ii) $\Lambda_1^\# (dH_0) = \Lambda_0^\# (dH_1)$, with

$$
H_0 = \sum_{i=1}^{n} \left( \exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right)
$$

and

$$
H_1 = \sum_{i=1}^{n} \left( \frac{1}{2} \left( \exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right)^2 + \exp(q^{i-1} - q^i + p_{i-1}) \left( \exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right) \right),
$$

where, by convention, $q^0 = -\infty$ and $q^{n+1} = +\infty$;

(iii) the map $\pi : (\mathbb{R}^{2n}, \Lambda_0) \rightarrow (\mathbb{R}^{2n-1}, \tilde{\Lambda}_0)$,

$$(q^1, \ldots, q^n, p_1, \ldots, p_n) \mapsto (c_1, \ldots, c_{n-1}, d_1, \ldots, d_n),$$

is a Poisson morphism.

**Proof.**

A simple computation leads to the required results. Relations (i) and (ii) prove the existence of a bi-Hamiltonian system on $\mathbb{R}^{2n}$, while (iii) ensures that $\Lambda_0$ reduces to $\tilde{\Lambda}_0$. $\square$

Since $\Lambda_0$ is non-degenerate, we can define a recursion operator, by setting $R = \Lambda_1^\# \circ (\Lambda_0^\#)^{-1}$. We get

$$
R = \begin{bmatrix}
A & B \\
C & A^T
\end{bmatrix}
$$

5
where \( A = [a_{ij}], B = [b_{ij}] \) and \( C = [c_{ij}] \) are \( n \times n \) matrices \((A^T \text{ is the transpose of } A)\), that are defined as follows (with the convention that \( q^{n+1} = +\infty \)):

\[
\begin{align*}
\alpha_{ii} &= \exp(q^i - q^{i+1} + p_i) \\
\alpha_{i,i+1} &= \exp(q^{i+1} - q^{i+2} + p_{i+1}) \\
\alpha_{ij} &= 0, \quad \text{if } i > j \\
\alpha_{ij} &= \exp(q^i - q^{i+1} + p_j) - \exp(q^{j-1} - q^j + p_{j-1}), \quad \text{if } j > i + 1;
\end{align*}
\]

and

\[
\begin{align*}
b_{ij} &= -b_{ji} \\
b_{ij} &= \exp(q^i - q^{i+1} + p_j) + \exp(p_j), \quad \text{if } i < j
\end{align*}
\]

and

\[
\begin{align*}
c_{i+1,i} &= -c_{i,i+1} = \exp(q^i - q^{i+1} + p_i) \\
c_{ij} &= 0, \quad \text{otherwise}.
\end{align*}
\]

Once we have a recursion operator \( R \), an infinite sequence of pairwise compatible Poisson tensors on \( \mathbb{R}^{2n} \), \( \Lambda_k = R^k \Lambda_0 \), and an infinite sequence of Hamiltonians \( H_k \) given by \( dH_k = iR_i(dH_{k-1}) \) are then defined. By the reduction theorem of bi-Hamiltonian manifolds [13], and taking account of 1.1(iii), the infinite sequence \((\Lambda_k), k \in \mathbb{N}_0\), of higher order Poisson tensors on \( \mathbb{R}^{2n} \) reduce, by \( \pi_2 \) to an infinite sequence \((\bar{\Lambda}_k), k \in \mathbb{N}_0\), of pairwise compatible Poisson tensors on \( \mathbb{R}^{2n-1} \).

For \( i = 2 \), the Poisson tensor \( \Lambda_2 \) is given by

\[
\Lambda_2 = \sum_{j=1}^{n-2} \frac{\partial}{\partial q^j} \wedge \left( \exp(q^{j+1} - q^{j+2} + p_{j+1}) \frac{\partial}{\partial p_{j+1}} \right) + \sum_{i=j+1}^{n-1} \left( -\exp(q^{i+1} - q^{i+2} + p_{i+1}) - \exp(p_{i+1}) \right) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_{i+1}} \right)
\]

\[
+ \sum_{i=1}^{n} \left( \exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right) \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} \right)
\]

\[
- \exp(q^i - q^{i+1} + p_i) \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial p_{i+1}} \right),
\]

and the corresponding reduced Poisson tensor \( \bar{\Lambda}_2 \) on \( \mathbb{R}^{2n-1} \) is the one associated with the cubic bracket that appears in [6] and also in [4].

2 Master symmetries for the relativistic Toda lattice

Now, we want to find master symmetries for the bi-Hamiltonian system built above, in order to use the method of R. L. Fernandes [7], which is based on the following theorem.
Theorem 2.1 (Oevel) Let $X_0$ be a vector field on the Poisson-Nijenhuis manifold $(M, \Lambda_0, \Lambda_1)$, such that

$$\mathcal{L}(X_0)\Lambda_0 = \alpha \Lambda_0, \quad \mathcal{L}(X_0)\Lambda_1 = \beta \Lambda_1 \quad \text{and} \quad X_0.H_0 = \gamma H_0,$$  

(10)

with $\alpha, \beta, \gamma \in \mathbb{R}$. Then the vector fields $X_k = R^k X_0$ satisfy, for all $k, l \in \mathbb{N}$,

(i) $[X_k, X_l] = (\beta - \alpha)(l - k)X_{k+l};$

(ii) $[X_k, Y_l] = (\beta + \gamma + (\beta - \alpha)(l - 1))Y_{k+l};$

(iii) $\mathcal{L}(X_k)\Lambda_l = (\beta + (\beta - \alpha)(l - k - 1))\Lambda_{k+l};$

(iv) $X_k.H_l = (\gamma + (\beta - \alpha)(l + k))H_{k+l}.$

We have to define a vector field that satisfies conditions (10) of theorem 2.1. We take

$$X_0 = \sum_{i=1}^n \frac{\partial}{\partial p_i},$$  

(11)

and we compute

$$\mathcal{L}(X_0)\Lambda_0 = -\Lambda_0, \quad \mathcal{L}(X_0)\Lambda_1 = 0 \quad \text{and} \quad X_0.H_0 = H_0.$$

So theorem 2.1 holds with $\alpha = -1, \beta = 0$ and $\gamma = 1$ and we have a hierarchy of master symmetries $X_k = R^k X_0, k \in \mathbb{N}$, that provides a way of getting higher order Poisson tensors on $\mathbb{R}^{2n}$. These master symmetries satisfy the following conditions:

\begin{enumerate}
  \item [a)] $[X_k, X_l] = (l - k)X_{k+l};$
  \item [b)] $[X_k, \Lambda_l^\#(dH_0)] = l\Lambda_{k+l}^\#(dH_0);$
  \item [c)] $\mathcal{L}(X_k)\Lambda_l = (l - k - 1)\Lambda_{k+l};$
  \item [d)] $X_k.H_l = (1 + l + k)H_{k+l}.$
\end{enumerate}

Proposition 2.1 The master symmetries $X_k = R^k X_0, k \in \mathbb{N}$, are projectable vector fields by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}, \quad (q^i, p_i) \mapsto (c_i, d_i)$. We denote by $\tilde{X}_k$ the projected vector fields.

Proof.

The fibres of $\pi$ are integral curves of the vector field

$$Z = \sum_{i=1}^n \frac{\partial}{\partial q^i}.$$

Since $Z = -\sum_{i=1}^n \Lambda_l^\#(dp_i)$, this implies $[\Lambda_1, Z] = 0$.

Further, we compute $[\Lambda_0, Z] = 0$ and therefore $[R, Z] = 0$. Also, $[X_0, Z] = 0$ and we deduce

$$[X_k, Z] = [R^k X_0, Z] = 0.$$
For any \( f \in C^\infty(\mathbb{R}^{2n}, \mathbb{R}) \), we have

\[
[X_k, f Z] = (X_k, f) Z + f [X_k, Z] = (X_k, f) Z
\]

which proves that \( X_k \) is a projectable vector field. \( \square \)

Now, if we take the reduced vector fields \( \tilde{X}_k \), the reduced Poisson tensors \( \tilde{\Lambda}_k \), the reduced Hamiltonian vector fields \( \tilde{Y}_k = \tilde{\Lambda}_k^\#(d\tilde{H}_0) \) and the reduced Hamiltonians \( \tilde{H}_k \) then, taking account of relations (12), (13), (14) and (15), we deduce the following relations:

a) \( [\tilde{X}_k, \tilde{X}_i] = (l - k)\tilde{X}_{k+i} \) \hspace{1cm} (16)

b) \( [\tilde{X}_k, \tilde{Y}_i] = l\tilde{Y}_{k+i} \) \hspace{1cm} (17)

c) \( \mathcal{L}(\tilde{X}_k)\tilde{\Lambda}_i = (l - k - 1)\tilde{\Lambda}_{k+i} \) \hspace{1cm} (18)

d) \( \tilde{X}_k.\tilde{H}_l = (1 + l + k)\tilde{H}_{k+l} \) \hspace{1cm} (19)

Some of these relations already appeared in [6], although our vector field \( \tilde{X}_1 \) differs from the corresponding one in [6] - let us denote it by \( \hat{X}_1 \), by a bi-Hamiltonian vector field. In fact, we compute

\[
X_1 = \sum_{i=1}^{n} \left( (1 - i) \left( \exp(q^i - q^{i+1} + p_i) + \exp(p_i) \right) \right.
+ \sum_{j=i+1}^{n} \left( \exp(q^j - q^{j+1} + p_j) + \exp(p_j) \right) \frac{\partial}{\partial q^i}
+ \sum_{i=1}^{n} \left( \exp(p_i) + i \exp(q^i - q^{i+1} + p_i) \right.
+ \left. (2 - i) \exp(q^{i-1} - q^i + p_{i-1}) \right) \frac{\partial}{\partial p_i},
\]

where, by convention, \( q^0 = -\infty \) and \( q^{n+1} = +\infty \), and \( X_1 \) projects onto

\[
\hat{X}_1 = \sum_{i=1}^{n} \left( \left( (1 + i)c_i(c_{i+1} + d_{i+1}) + (2 - i)c_i(c_{i-1} + d_i) + c_i^2 \right) \frac{\partial}{\partial c_i}
+ \left( i c_i d_i + (2 - i)c_{i-1} d_i + d_i^2 \right) \frac{\partial}{\partial d_i} \right),
\]

where, by convention, \( c_0 = c_n = 0 \). Comparing \( \tilde{X}_1 \) and \( \hat{X}_1 \), we obtain

\[
\hat{X}_1 - \tilde{X}_1 = \tilde{\Lambda}_1^\#(d\tilde{H}_0) = \tilde{\Lambda}_0^\#(d\tilde{H}_1).
\]

Since the difference is the Hamiltonian vector field \( \tilde{\Lambda}_1^\#(d\tilde{H}_0) \), our higher order Poisson tensors \( \tilde{\Lambda}_k \) coincide with those of [6].
References


