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POISSON MANIFOLDS IN MECHANICS

1. INTRODUCTION

Poisson structures were defined and studied by Lichnerowicz (1976, 1977), who recognized their importance in mechanics and mathematical physics. Under the name of Hamiltonian structures, several other authors gave various definitions of Poisson structures, equivalent to the definition used by Lichnerowicz: among others, we refer to Iacob and Sternberg (1979), Kuperschmidt and Manin (1977), Symes (1980 a and b). Poisson structures are in fact a particular case of local Lie algebras, studied by Kirillov (1974, 1976). In this introduction, we will indicate some of the reasons which account for the growing importance of Poisson structures in mechanics.

1.1. Poisson manifolds as reduced phase spaces of Hamiltonian systems

Classically (Abraham and Marsden, 1978, Arnold, 1974), a Hamiltonian mechanical system is mathematically described by a symplectic manifold (M,Ω), called the phase space of the system, and a differentiable function $H : M \to \mathbb{R}$, called the Hamiltonian of the system. The time evolution of the system is described by integral curves of the Hamiltonian vector field $\#dH$, defined by the property :

$$i(\#dH)\Omega = - dH \quad .$$

Let G be a Lie group of symmetries of the system, that means, a Lie group which acts on the manifold M, by an action which preserves the symplectic 2-form Ω and the Hamiltonian H; for all $g \in G$, we have :

$$g^{\star}\Omega = \Omega \quad ; \quad g^{\star}H = H \circ g = H \quad .$$

Let us assume that the set P of orbits of the G-action on M has a differentiable manifold structure, such that the

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canonical projection $\pi: M \to P$ is a submersion. The manifold P is called the reduced phase space of the system. It can be shown that the Hamiltonian vector field $\#dH$ projects onto M. For studying motions of the system, one may first look at integral curves of the projected vector field, on the reduced phase space P. But in general P is no more a symplectic manifold: it is a Poisson manifold.

A classical example of such a situation is the Euler-Poinsot motion of a rigid body with a fixed point (Arnold, 1974): the phase space is the cotangent bundle $T^\star(SO(3))$ to the rotation group $SO(3)$; the Hamiltonian H is a left-invariant Riemannian metric H on the group $SO(3)$, which may be defined by a symmetric map $I: \mathcal{G} \to \mathcal{G}^\star$, called the inerty operator; here $\mathcal{G}$ is the Lie algebra of the rotation group, and $\mathcal{G}^\star$ its dual space. The reduced phase space is $\mathcal{G}^\star$, and the projection on $\mathcal{G}^\star$ of the Hamiltonian vector field $\#dH$, leads to the Euler differential equation:

$$\frac{d\mu}{dt} = - \mathrm{ad}^\star_{I^{-1}(\mu)}\mu \quad , \quad (\mu \in \mathcal{G}^\star) \quad ,$$

where $\mathrm{ad}^\star$ stands for the coadjoint representation of $\mathcal{G}$.

Poisson structures on reduced phase spaces may also be encountered in more general situations, for instance when such a reduced phase space is the set of leaves of a foliation of a symplectic manifold, instead of the space of orbits of a Lie group action; see Proposition 2.10 and Example 2.11 in the following.

1.2. Poisson manifolds a spaces of values of momentum maps

As in 1.1, let (M,Ω,H) be a classical Hamiltonian system, and G a Lie group which acts on M by a Hamiltonian action; for each $g \in G$, we have:

$$g^\star\Omega = \Omega \quad ,$$

and moreover, there exists a differentiable map $J: M \to \mathcal{G}^\star$, such that for all $X \in \mathcal{G}$:

$$i(X_M)\Omega = - d\langle J \; , X\rangle \quad .$$

Here $\mathcal{G}$ is the Lie algebra of G, $\mathcal{G}^\star$ its dual space, and X_M the vector field on M defined by:

$$X_M(x) = \frac{d}{dt} \exp(-tX).x\big|_{t=0} \quad , \quad (x \in M).$$

The map J was first defined by Souriau (1969), and is called the momentum of the G-action. Souriau has shown that there exists an affine action of G on $\mathcal{G}^\star$:

$$a : G \times \mathcal{G}^\star \to \mathcal{G}^\star$$

$$a(g,\xi) = Ad^\star_g \xi + \theta(g)$$

(where θ: $G \to \mathcal{G}^\star$ is a symplectic 1-cocycle of G), for which the map J is equivariant. Moreover, there exists on $\mathcal{G}^\star$ a Poisson structure, called the modified Kirillov-Kostant-Souriau structure associated with the symplectic cocycle θ, for which J is a Poisson morphism (see example 2.3, 3°, and definition 2.8 below).

The Hamiltonian H is no more assumed G-invariant, but we assume that it may be written as:

$$H = \hat{H} \circ J \quad ,$$

where $\hat{H} : \mathcal{G}^\star \to \mathbb{R}$ is some differentiable function. Then it can be shown that the Hamiltonian vector field #dH on the symplectic manifold M, and the Hamiltonian vector field $\#d\hat{H}$ on the Poisson manifold $\mathcal{G}^\star$, (def. 2.6 below), are J-related.

An example of such a situation is the Euler-Lagrange motion of a rigid body in a gravity field, with a fixed point on its revolution axis. The phase space is $T^\star(SO(3))$, just as for the Euler-Poinsot motion. But the Lie group G used now is the group of displacements of a three-dimensional Euclidean space, $SO(3) \times \mathbb{R}^3$ (semi-direct product). See for instance Iacob and Sternberg (1979).

1.3. Remark

The two ways by which Poisson structures appear, described in 1.1 and 1.2, are related by the following fact: under suitable assumptions, the set of values of the momentum map J appears as the set of leaves of the generalized foliation of the manifold M defined by the symplectic orthogonal of the set of subspaces tangent to orbits of the G-action. This property seems closely related to example 2.11 given below.

1.4. Poisson and canonical manifolds

For the mathematical description of Hamiltonian mechanical systems with time-dependent Hamiltonians and constraints, Lichnerowicz (1976, 1979) has defined and studied canonical manifolds, which are Poisson manifolds with an additionnal structure. For more details on this subject, the reader is referred to the papers of Lichnerowicz quoted above, and to Marle (1982).

1.5. Poisson manifolds and completely integrable systems

A considerable interest was raised up recently by completely integrable Hamiltonian systems: see the works of Lax (1968), Adler (1979), Adler and Van Moerbeke (1980, a and b), Iacob and Sternberg (1979), Kazhdan, Kostant and Sternberg (1978), Kostant (1979), Mischenko and Fomenko (1978), Moser (1975), Olshanetsky and Perelomov (1976, 1979), Ratiu (1980), Reyman and Semenov-Tian-Shansky (1979, 1981), Symes (1980, a and b); see also the conference of Verdier (1980) at the Séminaire Bourbaki. In these works, Poisson structures appear; they are mainly of Kirillov-Kostant-Souriau type, and defined on dual spaces of Lie algebras. As will be seen in paragraph 5 below, some of the involution theorems obtained in these works may be put under a simpler and more general form, by the use of general Poisson structures.

2. POISSON MANIFOLDS: ELEMENTARY PROPERTIES AND EXAMPLES

2.1. Definition

A Poisson structure on a differential manifold M, is defined by a bilinear map :

$$C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R}) \quad ,$$

called Poisson bracket, and noted : $(f,g) \mapsto \{f,g\}$, satisfying the following properties:

i) the Poisson bracket is skew-symmetric:

$$\{g,f\} = - \{f,g\} \quad ;$$

ii) it is a derivation in each of its arguments :

$$\{f_1 f_2, g\} = \{f_1, g\} f_2 + f_1 \{f_2, g\} \quad ;$$

$$\{f, g_1 g_2\} = \{f, g_1\} g_2 + g_1 \{f, g_2\} \quad ;$$

iii) it satisfies the Jacobi identity :

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0 \quad .$$

With such a structure, the manifold M is called a Poisson manifold.

2.2. Remark

The space $C^\infty(M,\mathbb{R})$ of differentiable functions on a Poisson manifold M, is endowed with two algebraic structures:
an associative algebra structure, defined by the ordinary product $(f,g) \mapsto fg$;
a Lie algebra structure, defined by the Poisson bracket $(f,g) \mapsto \{f,g\}$.

These two structures are related by the property ii) of definition 2.1, which may be put under the following form : for any $f \in C^\infty(M,\mathbb{R})$, let ad_f be the linear endomorphism of $C^\infty(M,\mathbb{R})$:

$$g \mapsto \mathrm{ad}_f(g) = \{f,g\} \quad ;$$

then, ad_f is a derivation of the associative algebra structure defined by the ordinary product.

A real vector space with an associative algebra structure and a Lie algebra structure related in such a way, will be called a Poisson algebra. Many properties of Poisson manifolds are in fact properties of the corresponding Poisson algebra, and remain valid for any Poisson algebra. This idea is developed by Ouzilou (1981).

2.3. Examples

1°) Let (M,Ω) be a symplectic manifold. The 2-form Ω defines an isomorphism :

$$(\#) : T^\star M \to TM \quad ;$$

by definition, for all $x \in M$, $\alpha \in T^\star_x M$, $\#\alpha$ is the unique vector of T_xM such that :

$$i(\#\alpha)\ \Omega = -\ \alpha \quad .$$

The Poisson bracket associated with the symplectic structure on M, is the bilinear map from $C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R})$ into $C^\infty(M,\mathbb{R})$:

$$(f,g) \mapsto \{f,g\} = \#df.g = -\ \#dg.f = \Omega(\#df\ ,\ \#dg) \quad .$$

One may check that this Poisson bracket satisfies the properties of Definition 2.1. This shows that any symplectic manifold has an underlying Poisson structure.

2°) Let $\mathcal{G}$ be a real, finite dimensional Lie algebra; the bracket of two elements X and Y of $\mathcal{G}$ will be noted $[X,Y]$. Let $\mathcal{G}^\star$ be the dual space of $\mathcal{G}$. For all f and $g \in C^\infty(\mathcal{G}^\star,\mathbb{R})$, and all $x \in M$, we set :

$$\{f,g\}(x) = \langle x\ ,\ [df(x)\ ,\ dg(x)]\rangle \quad .$$

One may check that this Poisson bracket satisfies the properties of Definition 2.1. This proves that the dual space $\mathcal{G}^\star$ of a real, finite dimensional Lie algebra $\mathcal{G}$, has a natural Poisson structure. This structure was defined by A. Kirillov (1974), B. Kostant (1970), and J.-M. Souriau (1969).

3°) With the same hypotheses and notations as in the last example, let $\Theta : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ be a skew-symmetric bilinear form on $\mathcal{G}$, such that, for all X, Y and $Z \in \mathcal{G}$:

$$(\star) \qquad \Theta(X\ ,\ [Y,Z]) + \Theta(Y\ ,\ [Z,X]) + \Theta(Z\ ,\ [X,Y]) = 0 \quad .$$

We will say that Θ is a symplectic 2-cocycle of $\mathcal{G}$ (with values in $\mathbb{R}$), or, when Θ is looked at as a linear map from $\mathcal{G}$ into its dual space $\mathcal{G}^\star$, a symplectic 1-cocycle of $\mathcal{G}$ with values in $\mathcal{G}^\star$.

For all f and $g \in C^\infty(\mathcal{G}\ ,\ \mathbb{R})$ and $x \in M$, we set :

$$\{f,g\}_\Theta(x) = \langle x\ ,\ [df(x)\ ,\ dg(x)]\rangle - \Theta(df(x)\ ,\ dg(x)) \quad .$$

One may check again that this Poisson bracket satisfies the properties of Definition 2.1. The corresponding Poisson structure on $\mathcal{G}^\star$ will be called the modified Kirillov-Kostant-

Souriau structure associated with the symplectic cocycle Θ. When $\Theta = 0$, it reduces to the Kirillov-Kostant-Souriau structure defined in the preceding example.

2.4. Proposition

Let M be a Poisson manifold. There exists on M a unique two times contravariant, skew-symmetric tensor field Λ such that, for all f and $g \in C^\infty(M, \mathbb{R})$:

$$(\star) \qquad \{f,g\} = \Lambda(df, dg) \quad .$$

The tensor field Λ is called the Poisson tensor field of M.

Proof. The defining property $(\star)$ shows that Λ is unique. For proving its existence, it is sufficient to check that for all f and $g \in C^\infty(M, \mathbb{R})$ and all $x \in M$, $\{f,g\}(x)$ depends only on $df(x)$ and $dg(x)$. But this results from property ii) in Definition 2.1.

2.5. Remark

Let Λ be a two times contravariant, skew-symmetric tensor field on a manifold M. The formula $(\star)$ in Proposition 2.4 defines a bilinear map $(f,g) \mapsto \{f,g\}$, from $C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R})$ into $C^\infty(M,\mathbb{R})$. This map satisfies properties i) and ii) of Definition 2.1, but in general it does not satisfy property iii). Lichnerowicz (1977) has shown that this map satisfies property iii), if and only if the tensor field Λ is such that:

$$(\star\star) \qquad [\Lambda, \Lambda] = 0 \quad ,$$

the bracket in this formula being the Schouten bracket, (Schouten 1954).

This shows that a Poisson structure on a manifold may be defined by a two times contravariant, skew-symmetric tensor field Λ, which satisfies property $(\star\star)$ above. This is the definition of Poisson structures initially introduced by Lichnerowicz (1977); it is equivalent to Definition 2.1.

2.6 Definition

Let M be a Poisson manifold, and Λ its Poisson tensor field. We will note (#) the morphism, from the cotangent bundle

$T^{\star}M$ into the tangent bundle TM, which associates with any $x \in M$ and $\alpha \in T_x^{\star}M$, the unique vector $\#\alpha \in T_xM$ such that, for all $\beta \in T_x^{\star}M$:

$$\langle\beta , \#\alpha\rangle = \Lambda_x(\alpha , \beta) \quad .$$

Let $f \in C^\infty(M , \mathbb{R})$. The vector field $\#df$ will be called the Hamiltonian vector field with Hamiltonian function f. It is characterized by the fact that, for any $g \in C^\infty(M , \mathbb{R})$:

$$\#df \,.\, g = \{f,g\} \quad .$$

2.7. Proposition

Let M be a Poisson manifold. The map :

$$f \mapsto \#df$$

is a Lie algebra homomorphism of $C^\infty(M , \mathbb{R})$ (with the Lie algebra structure defined by the Poisson bracket), into the space $\mathscr{C}(M)$ of C^∞ vector fields on M (with the Lie algebra structure defined by the usual bracket).

Proof. We must check that, for all f and $g \in C^\infty(M , \mathbb{R})$:

$$\#d\{f,g\} = [\#df , \#dg] \quad .$$

But for any third element h of $C^\infty(M , \mathbb{R})$, we have :

$$\begin{aligned}\#d\{f,g\}.h &= \{\{f,g\} , h\} = \{f , \{g,h\}\} - \{g , \{f,h\}\} \\ &= \#df.(\#dg.h) - \#dg.(\#df.h) = [\#df , \#dg].\, h\end{aligned}$$

2.8. Definition

Let M and N be two Poisson manifolds. A differentiable map $\varphi : M \to N$ is called a Poisson morphism if it is such that, for all f and $g \in C^\infty(N , \mathbb{R})$:

$$\varphi^{\star}\{f,g\} = \{\varphi^{\star}f , \varphi^{\star}g\} \quad .$$

The following proposition indicates a usuful property of Poisson morphisms.

2.9. Proposition

Let M and N be two Poisson manifolds, and $\varphi : M \to N$ a Poisson morphism. Then for any $f \in C^\infty(N,\mathbb{R})$, the pair of Hamiltonian vector fields $\#d(f_o\varphi)$ on M, and $\#df$ on N, is compatible with the map φ; this means that, for all $x \in M$:

$$T_x\varphi(\#d(f_o\varphi)(x)) = \#df(\varphi(x)) \quad .$$

Proof. Let g be another element of $C^\infty(N,\mathbb{R})$. We have:

$$\begin{aligned}
<dg(\varphi(x)), T_x\varphi(\#d(f_o\varphi)(x))> &= <d(g_o\varphi)(x) \ , \#d(f_o\varphi)(x)> \\
&= \{f_o\varphi \ , g_o\varphi\}(x) \\
&= \{\varphi^\star f \ , \varphi^\star g\}(x) \\
&= \varphi^\star\{f,g\}(x) = \{f,g\}(\varphi(x)) \\
&= <dg(\varphi(x) \ , \#df(\varphi(x))> \quad .
\end{aligned}$$

2.10. Proposition

Let M be a Poisson manifold, and $\varphi : M \to N$ be a surjective submersion of M onto a differentiable manifold N. The two following properties are equivalent.

1. For all f and $g \in C^\infty(M,\mathbb{R})$, the function $\{f_o\varphi \ , g_o\varphi\}$ is constant on any fiber of the fibration $\varphi : M \to N$.

2. There exists a Poisson structure on N such that φ is a Poisson morphism.

When these two equivalent properties are satisfied, the Poisson structure on N for which φ is a Poisson morphism is unique.

Proof. As φ is a surjective submersion, the map:

$$f \mapsto \varphi^\star f = f_o\varphi$$

is a vector space isomorphism of $C^\infty(N,\mathbb{R})$ onto the vector subspace of $C^\infty(M,\mathbb{R})$ of functions which are constant on each fiber of the fibration $\varphi : M \to N$. This shows that Property 2 implies Property 1. Conversely, if we assume Property 1 satisfied, we can define the Poisson bracket $\{f,g\}$ of two functions f and $g \in C^\infty(N,\mathbb{R})$ as the unique function on N such that:

$$\{f,g\} \circ \varphi = \{f \circ \varphi \,,\, g \circ \varphi\} \quad .$$

We can check that properties of Definition 2.1 are satisfied. At last the uniqueness of the Poisson structure on N for which φ is a Poisson morphism, is a consequence of Definition 2.8.

2.11. Example

This example is due to P. Libermann (1982). Let (M,Ω) be a symplectic manifold. We first recall some definitions and notations (see for instance Abraham and Marsden, 1978). If x is a point of M and F_x a vector subspace of the tangent space T_xM, the symplectic orthogonal of F_x is the vector subspace of T_xM :

$$\operatorname{orth} F_x = \{v \in T_xM \mid \Omega(x)(v,w) = 0 \text{ for all } w \in F_x\} \quad .$$

The vector space F_x is said coisotropic (resp. isotropic, resp. Lagrangian) if $\operatorname{orth} F_x \subset F_x$ (resp. if $F_x \subset \operatorname{orth} F_x$, resp. if $F_x = \operatorname{orth} F_x$).

Similarly, let F be a vector subbundle of TM. The symplectic orthogonal orth F of F is the vector subbundle of TM, whose fiber, at each point x of TM, is the symplectic orthogonal $\operatorname{orth} F_x$ of the fiber F_x of F at point x. The vector subbundle F is said coisotropic (resp. isotropic, resp. Lagrangian) if $\operatorname{orth} F \subset F$ (resp. if $F \subset \operatorname{orth} F$, resp. if $F = \operatorname{orth} F$).

We now consider the Poisson structure on M underlying its symplectic structure (example 2.3, 1°). Let $\varphi : M \to N$ be a surjective submersion of M onto a differentiable manifold N such that, for each $x \in N$, $\varphi^{-1}(x)$ is connected. The kernel $\ker(T\varphi)$ of the fibre bundle map $T\varphi : TM \to TN$ is a completely integrable vector subbundle of TM, and the manifold N may be looked at as the manifold of leaves of the foliation of M defined by $\ker(T\varphi)$. Then the two equivalent properties of Proposition 2.10 are satisfied if and only if the vector subbundle $\operatorname{orth}(\ker(T\varphi))$ of TM is completely integrable. This property follows from the Frobenius theorem, and from Proposition 2.7.

In particular, when $\ker(T\varphi)$ is a coisotropic subbundle of TM, it can be shown that $\mathrm{orth}(\ker(T\varphi))$ is completely integrable; therefore in that case, the two equivalent properties of Proposition 2.10 are satisfied.

2.12. Remark

Under the hypotheses of the last example, let Λ be the Poisson tensor field of N, defining the Poisson structure on N for which $\varphi : M \to N$ is a Poisson morphism. Let x be a point of N, and $2p$ the rank of Λ at point x. Then it can be seen that the rank of the 2-form induced on the submanifold $\varphi^{-1}(x)$ by the symplectic 2-form Ω, is constant, equal to $2(p+m-n)$, where $2m$ is the dimension of M, and n the dimension of N.

3. CHARACTERISTIC FIELD AND INTEGRAL MANIFOLDS OF A POISSON STRUCTURE

In this paragraph M is a Poisson manifold, and Λ its Poisson tensor field. For simplicity, in the following definitions and propositions, all functions, differential forms and vector fields considered are defined on the whole manifold M. The reader will check easily that the results can be extended to the case when these functions, differential forms or vector fields, are defined on open subsets of the manifold M.

3.1. Definitions

1°) A function $f \in C^\infty(M,\mathbb{R})$ is an invariant of the Poisson structure if it is an element of the center of the Lie algebra, that means, if for all $g \in C^\infty(M,\mathbb{R})$, one has :

$$\{f,g\} = 0 \quad .$$

2°) The characteristic field of the Poisson structure is the subset $\mathcal{F}$ of the tangent bundle TM, image of the fibre bundle morphism $(\#) : T^\star M \to TM$.

3°) An integral manifold of the Poisson structure is a connected, immersed submanifold N of M such that, for all $x \in N$:

$$T_xN = \mathcal{F}_x \quad ,$$

where $\mathcal{F}_x$ is the fiber at point x of the characteristic field $\mathcal{F}$.

3.2. Remarks

1°) A function $f \in C^\infty(M,\mathbb{R})$ is an invariant of the Poisson structure if and only if its differential df is a section of the annihilator $\mathcal{F}^0$ of $\mathcal{F}$, that means, if and only if, for each $x \in M$, $df(x)$ belongs to the vector subspace of $T_x^\star M$ of linear forms on T_xM which vanish on the vector subspace $\mathcal{F}_x$.

2°) For each $x \in M$, the fiber $\mathcal{F}_x$ of the characteristic field at point x , is a vector subspace of T_xM , whose dimension is equal to the rank of the skew-symmetric two times contravariant tensor $\Lambda(x)$. But in general, the dimension of $\mathcal{F}_x$ depends on the point $x \in M$; for this reason, $\mathcal{F}$ is not always a vector subbundle of TM.

3.3. Proposition

Let α and β be two Pfaff forms on M, of class C^∞. There exists a Pfaff form γ on M, of class C^∞, such that:

$$[\#\alpha\ ,\ \#\beta] = \#\gamma\quad .$$

Proof. The use of a partition of unity enables us to treat the problem locally, in the domain U of a chart of the manifold M. Let $x^1, \dots x^m$ be the local coordinates associated with this chart. The Pfaff forms α and β may be written locally as:

$$\alpha = \sum_{i=1}^{m} \alpha_i\ dx^i \quad ; \quad \beta = \sum_{j=1}^{m} \beta_j\ dx^j \quad ,$$

where α_i and β_j are C^∞ functions defined on U. We have:

$$\#\alpha = \sum_{i=1}^{m} \alpha_i\ \#dx^i \quad ; \quad \#\beta = \sum_{j=1}^{m} \beta_j\ \#dx^j \quad ;$$

therefore:

$$[\#\alpha\ ,\ \#\beta] = \sum_{i=1}^{m} \sum_{j=1}^{m} [\alpha_i\ \#dx^i\ ,\ \beta_j\ \#dx^j]$$

or, using well known formulas about the bracket :

$$[\#\alpha , \#\beta] = \sum_{i=1}^{m} \sum_{j=1}^{m} (\alpha_i \beta_j [\#dx^i , \#dx^j]$$

$$+ \alpha_i \langle d\beta_j , \#dx^i \rangle \#dx^j$$

$$- \beta_j \langle d\alpha_i , \#dx^j \rangle \#dx^i) .$$

By using Proposition 2.7, we have:

$$[\#dx^i , \#dx^j] = \#d\{x^i , x^j\} ,$$

and we have:

$$[\#\alpha , \#\beta] = \#\gamma ,$$

where:

$$\gamma = \sum_{i=1}^{m} \sum_{j=1}^{m} (\alpha_i \beta_j d\{x^i,x^j\} + \alpha_i (\#dx^i.\beta_j) dx^j - \beta_j (\#dx^j.\alpha_i) dx^i).$$

The last proposition shows that the space of sections of $\mathcal{F}$ which are images, by the fibre bundle morphism (#), of C^∞ Pfaff forms on M, is invariant by the bracket operation. This property looks like the Frobenius condition, for the complete integrability of a vector subbundle of the tangent bundle. But the classical Frobenius theorem is not applicable here, because $\mathcal{F}$ is not a vector subbundle of TM. However, we have the following result, due to A. Kirillov (1976):

3.4. <u>Theorem</u>

Let x be any point of the Poisson manifold M. There exists a unique maximal integral manifold N_x of the Poisson structure containing the point x. Any other integral manifold of the Poisson structure which contains x, is a connected, open submanifold of N_x. Moreover, N_x has a unique symplectic structure, whose symplectic 2-form is defined by the following property, valid for all f and g $\in C^\infty(M,\mathbb{R})$ and all y $\in N_x$:

$$(\star) \qquad \Omega_{N_x}(y) \, (\#df(y) , \#dg(y)) = \{f,g\}(y) .$$

Finally, the manifold M is partitioned into maximal integral manifolds of its Poisson structure, which are symplectic connected immersed submanifolds of M (in general, not all of the same dimension).

Proof. We will first prove the existence of a local integral manifold N of the Poisson structure containing x. Let 2p be the dimension of $\mathcal{F}_x$. If $p = 0$, the result is true, because $N_x = \{x\}$. We assume now $p > 0$; therefore, there exists a function $f \in C^\infty(M,\mathbb{R})$ such that $\#df(x) \neq 0$. By integration along integral curves of $\#df$, we can define, on an open neighbourhood U of x, a differentiable function g such that:

$$\#df.g = 1 \quad ,$$

or, according to the very definition of $\#df$:

$$\{f,g\} = 1 \quad .$$

By Proposition 2.7, we have on U:

$$[\#df \ , \#dg] = 0 \quad .$$

Moreover, the vector fields $\#df$ and $\#dg$ are linearly independent at each point of U: if a and b are two scalars and y a point of U such that:

$$a \ \#df(y) + b \ \#dg(y) = 0 \quad ,$$

we have:

$$\#d(af + bg)(y) = 0 \quad ,$$

and:

$$\begin{aligned} 0 = \#d(af + bg).(-bf + ag) \ (y) &= \{af + bg \ , -bf + ag\}(y) \\ &= a^2 + b^2 \quad , \end{aligned}$$

which shows that $a = b = 0$.
By restricting eventually U, we may assume that there exists a surjective submersion:

$$\varphi \ : U \to W$$

of U onto a manifold W, whose dimension is dim M − 2, such that each leaf of the foliation of U defined by $\#df$ and $\#dg$ is the inverse image by φ of a point of W. Let h_1 and h_2 be two differentiable functions defined on U, which are constant on each fiber of the fibration $\varphi : U \to W$. We have:

$$\#df.h_i = 0 \quad ; \quad \#dg.h_i = 0 \quad , \quad (i = 1 \text{ or } 2);$$

$$\#df.\{h_1,h_2\} = \{f\ ,\ \{h_1,h_2\}\} = \#dh_1.(\#df.h_2) - \#dh_2.(\#df.h_1)$$
$$= 0 \quad ,$$

and similarly:

$$\#dg.\{h_1,h_2\} = 0 \ .$$

This shows that $\{h_1,h_2\}$ is constant on each fiber of the fibration $\varphi : U \to W$. By Proposition 2.10, we see that there exists on W a unique Poisson structure for which $\varphi : U \to W$ is a Poisson morphism. The rank of the Poisson tensor field Λ_W at point $\varphi(x)$, is equal to $2(p-1)$. If $p-1 = 0$, $\varphi^{-1}(\varphi(x))$ is an integral manifold of the Poisson structure containing x; the rank of the Poisson tensor field Λ is constant along integral curves of the vector fields $\#df$ and $\#dg$, because the integral flows of these vector fields are one-parameter local groups of Poisson morphisms; therefore the rank of Λ is constant along $\varphi^{-1}(\varphi(x))$, and equal to 2, that means, equal to the dimension of this manifold. This shows that $\varphi^{-1}(\varphi(x))$ satisfies the condition defining integral manifolds of the Poisson structure.
Now if $p-1 > 0$, replacing M by W, we can repeat the same argument as above. After a finite number of steps, we can assert the existence of a 2p-dimensional integral manifold N of the Poisson structure, containing the point x.
If N' is another 2p-dimensional integral manifold of the Poisson structure containing x, we see that $N \cap N'$ is open in N and in N'. This shows the local uniqueness of N. Then the existence and uniqueness of a maximal integral manifold of the Poisson structure containing x, is proved by the same procedure as in the case of a foliation of M (Chevalley 1946).
At last, we check that the 2-form Ω_{N_x} defined on N_x by the formula (★) above, is of class C^∞; it is non degenerate by the very definition of $\mathfrak{F}$, and closed because we have, if f, g and h are three differentiable functions, and y a point of N_x:

$$\begin{aligned}
d\Omega_{N_x}(\#df\,,\,\#dg\,,\,\#dh)(y) &= \underset{(f,g,h)}{\mathfrak{S}} (\#df.\Omega_{N_x}(\#dg\,,\,\#dh))(y) \\
&\quad - \underset{(f,g,h)}{\mathfrak{S}} \Omega_{N_x}([\#df\,,\,\#dg]\,,\,\#dh)(y) \\
&= \underset{(f,g,h)}{\mathfrak{S}} \{f\,,\,\{g,h\}\}(y) \\
&\quad - \underset{(f,g,h)}{\mathfrak{S}} \{\{f,g\}\,,\,h\}(y) \\
&= 0\,,
\end{aligned}$$

by 2.7 and the Jacobi identity. The symbol $\underset{(f,g,h)}{\mathfrak{S}}$ in the formulae above stands for a sum over the three circular permutations of (f,g,h).

3.5. Examples.

1°) In example 2.3, 1°, when (M,Ω) is a symplectic manifold, we have $\mathcal{F} = TM$; therefore the maximal integral manifolds of the Poisson structure are the connected components of M. More generally, when for all $x \in M$, the dimension of $\mathcal{F}_x$ is an even integer 2p which does not depend on x, $\mathcal{F}$ is a completely integrable vector subbundle of TM, whose rank is 2p. The maximal integral manifolds of the Poisson structure are the leaves of the foliation of M defined by $\mathcal{F}$: they are symplectic manifolds, all of the same dimension 2p. In that case, one can prove a version of Darboux theorem (Lichnerowicz 1977, Symes 1980): every point of M has an open neighbourhood, domain of a chart with local coordinates $x^1, \ldots x^{2p}, x^{2p+1}, \ldots x^m$, such that the expression of the Poisson bracket of two functions f and g is

$$\{f,g\} = \sum_{i=1}^{p} \left(\frac{\partial f}{\partial x^{p+i}}\frac{\partial g}{\partial x^i} - \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^{p+i}}\right) .$$

2°) We consider now example 2.3, 2°, when M is the dual space of a real, finite dimensional Lie algebra $\mathcal{G}$. Let G be a connected Lie group having $\mathcal{G}$ as Lie algebra. Any element X of $\mathcal{G}$ may be looked at as a linear function on $\mathcal{G}^\star$. The corresponding Hamiltonian vector field $\#dX$ on $\mathcal{G}^\star$ must satisfy, for all $Y \in \mathcal{G}$ and all $x \in \mathcal{G}^\star$ (the second equality below being a consequence of the very definition of the coadjoint representation $X \mapsto ad_X$ of the Lie algebra $\mathcal{G}$):

$$<\#dX(x) \; , \; Y> \; = \; <x \; , \; [X,Y]> \; = \; <-ad^{\star}_X \, x \; , \; Y> \quad .$$

Let $\gamma \mapsto Ad_\gamma$ and $\gamma \mapsto Ad^{\star}_\gamma$ (with $\gamma \in G$) be the adjoint and coadjoint representations of the Lie group G. We recall that, for all $\gamma \in G$, $x \in \mathcal{G}^{\star}$, $Y \in \mathcal{G}$:

$$<Ad^{\star}_\gamma \, x \; , \; Y> \; = \; <x \; , \; Ad_{\gamma^{-1}} \, Y> \quad ,$$

and that, for all $X \in \mathcal{G}$, $x \in \mathcal{G}^{\star}$, $Y \in \mathcal{G}$:

$$\frac{d}{dt} Ad^{\star}_{\exp(tX)} \, x \Big|_{t=0} = ad^{\star}_X \, x \quad ;$$

$$\frac{d}{dt} Ad_{\exp(tX)} \, Y \Big|_{t=0} = [X,Y] \quad .$$

We can see now that the maximal integral manifold N_x of the Poisson structure containing a point x of $\mathcal{G}^{\star}$ is the orbit through x of the coadjoint representation of G :

$$N_x = \{ Ad^{\star}_\gamma \, x \mid \gamma \in G \} \quad .$$

3°) With the same notations as above, we consider now the case of example 2.3, 3°, when the Poisson structure on $\mathcal{G}^{\star}$ is the modified Kirillov-Kostant-Souriau Poisson structure, associated with a symplectic cocycle Θ. It can be shown that there exists a unique differentiable map $\theta : G \to \mathcal{G}^{\star}$, which has the two properties :

i) for all γ_1 and $\gamma_2 \in G$, one has :

$$\theta(\gamma_1\gamma_2) = Ad^{\star}_{\gamma_1} \, \theta(\gamma_2) + \theta(\gamma_1) \quad ;$$

ii) if $T_e\theta : \mathcal{G} \to \mathcal{G}^{\star}$ is the linear map tangent to θ at the neutral element e of G, one has for all X and $Y \in \mathcal{G}$:

$$<T_e\theta(X) \; , \; Y> \; = \; \Theta(X,Y) \quad .$$

We will say that θ is a symplectic 1-cocycle of the Lie group G, with values in $\mathcal{G}^{\star}$, associated with the symplectic cocycle Θ of the Lie algebra $\mathcal{G}$.

Associated with θ, there exists an affine action a_θ of the Lie group G on the dual space $\mathcal{G}^{\star}$ of its Lie algebra, whose linear part is the coadjoint action, defined by :

$$a_\theta(\gamma,x) = \mathrm{Ad}^\star_\gamma\, x + \theta(\gamma) \qquad (\gamma \in G,\ x \in \mathfrak{G}^\star) \quad .$$

By the same procedure as in case 2° above, we can now see that the maximal integral manifold N_x of the Poisson structure containing a point x of $\mathfrak{G}^\star$, is the orbit through x of the affine action a_θ :

$$N_x = \{\mathrm{Ad}^\star_\gamma\, x + \theta(\gamma) \mid \gamma \in G\} \quad .$$

3.6. Remark (Symes 1980)

Under the hypotheses of examples 2.3, 3°, and 3.5, 3°, we consider the cotangent bundle $T^\star G$. The differential $d\alpha$ of the Liouville 1-form α, is a symplectic 2-form on $T^\star G$. The symplectic cocycle Θ may be looked at as a left invariant differential 2-form on the Lie group G, and the formula ($\star$) of example 2.3, 3°, shows that this 2-form is closed. Let :

$$\Omega_\Theta = d\alpha + q^\star\Theta \quad ,$$

where $q : T^\star G \to G$ is the canonical projection. One can check that Ω_Θ is a symplectic 2-form on $T^\star G$.

Let $\varphi : T^\star G \to \mathfrak{G}^\star$ be the map which associates, to each element z of $T^\star G$, the left invariant 1-form whose value at point $q(z) \in G$, is z. We see that φ is a surjective submersion, and that, for all $\xi \in \mathfrak{G}^\star$, $\varphi^{-1}(\xi)$ is the graph of the left invariant 1-form ξ. We are now under conditions of example 2.11: the modified Kirillov-Kostant-Souriau Poisson structure on $\mathfrak{G}^\star$ associated with the symplectic cocycle Θ, is the unique Poisson structure on $\mathfrak{G}^\star$ for which $\varphi : T^\star G \to \mathfrak{G}^\star$ is a Poisson morphism (when the Poisson structure on $T^\star G$ is the structure associated with the symplectic structure defined by Ω_Θ).

The above remark applies to examples 2.3, 2° and 3.5, 2° , by making $\Theta = 0$.

3.7. Remark

A differentiable function f defined on a Poisson manifold M is an invariant of the Poisson structure (definition 3.1), if and only if f is constant on each maximal integral manifold of the Poisson structure.

4. AUTOMORPHISMS AND INFINITESIMAL AUTOMORPHISMS OF A POISSON STRUCTURE

In this paragraph M is a Poisson manifold, and Λ its Poisson tensor field.

4.1. Definition

1°) A Poisson automorphism of M is a diffeomorphism $\varphi : M \to M$, which is also a Poisson morphism.

2°) An infinitesimal Poisson automorphism of M is a vector field X on M, whose integral flow φ is such that, for any $t \in \mathbb{R}$, φ_t is a Poisson morphism (from the open subset of M on which φ_t is defined, onto its image).

We can check that when φ and φ' are Poisson automorphisms of M, φ^{-1} and $\varphi' \circ \varphi$ are Poisson automorphisms of M : the set of Poisson automorphisms of M is a subgroup of the group of diffeomorphisms of M.

4.2. Examples

1°) Let (M,Ω) be a symplectic manifold; we look at M as a Poisson manifold, for the underlying Poisson structure. A diffeomorphism $\varphi : M \to M$ is a Poisson automorphism if and only if φ is a symplectomorphism of M, that means, if and only if :

$$\varphi^{\star}\Omega = \Omega \quad .$$

Under the same hypotheses, a vector field X on M is a Poisson infinitesimal automorphism, if and only if X is a locally Hamiltonian vector field on M (see for instance Abraham and Marsden, 1978), that means, if and only if the differential 1-form $i(X)\Omega$ is closed.

2°) Under the hypotheses of examples 2.3,3°, and 3.5, 3°, let γ be an element of the Lie group G. The affine transform of $\mathfrak{G}^{\star}$:

$$x \mapsto \varphi_\gamma(x) = a_\theta(\gamma,x) = \mathrm{Ad}^\star_\gamma\, x + \theta(\gamma)$$

is a Poisson automorphism of $\mathfrak{G}^\star$. We have indeed, for f and $g \in C^\infty(\mathfrak{G}^\star , \mathbb{R})$:

$$\begin{aligned}
\{\varphi^\star_\gamma f \, , \varphi^\star_\gamma g\}_\Theta(x) &= \langle x \, , [d(\varphi^\star_\gamma f)(x) \, , d(\varphi^\star_\gamma g)(x)]\rangle \\
&\quad - \Theta(d(\varphi^\star_\gamma f)(x) \, , d(\varphi^\star_\gamma g)(x)) \\
&= \langle x \, , [\mathrm{Ad}_{\gamma^{-1}}\, df(\varphi_\gamma(x)) \, , \mathrm{Ad}_{\gamma^{-1}}\, dg(\varphi_\gamma(x))]\rangle \\
&\quad - \Theta(\mathrm{Ad}_{\gamma^{-1}}\, df(\varphi_\gamma(x)) \, , \mathrm{Ad}_{\gamma^{-1}}\, dg(\varphi_\gamma(x))) \\
&= \langle \mathrm{Ad}^\star_\gamma\, x + \theta(\gamma) \, , [df(\varphi_\gamma(x)) \, , dg(\varphi_\gamma(x))]\rangle \\
&\quad - \Theta(df(\varphi_\gamma(x)) \, , dg(\varphi_\gamma(x))) \\
&= \varphi^\star_\gamma\{f,g\}\,(x) \quad .
\end{aligned}$$

In the above calculation we have used the property, valid for any $y \in \mathfrak{G}^\star$:

$$\begin{aligned}
\langle y \, , d(\varphi^\star_\gamma f)(x)\rangle &= \langle \mathrm{Ad}^\star_\gamma\, y \, , df(\varphi_\gamma(x))\rangle \\
&= \langle y \, , \mathrm{Ad}_{\gamma^{-1}}\, df(\varphi_\gamma(x))\rangle \quad .
\end{aligned}$$

We have also used the property which relates the cocycles θ of the Lie group G, and Θ of the Lie algebra $\mathfrak{G}$, valid for any X and $Y \in \mathfrak{G}$, and any $\gamma \in G$:

$$\langle\theta(\gamma) \, , [X,Y]\rangle = \Theta(X,Y) - \Theta(\mathrm{Ad}_{\gamma^{-1}}\, X \, , \mathrm{Ad}_{\gamma^{-1}}\, Y) \quad .$$

4.3. <u>Proposition</u>

Let X be a vector field on the Poisson manifold M. The three following properties are equivalent.

1°) For all f and $g \in C^\infty(M , \mathbb{R})$, we have :

$$X.\{f,g\} = \{X.f \, , g\} + \{f \, , X.g\} \quad .$$

2°) The Lie derivative of the Poisson tensor field Λ, with respect to the vector field X, vanishes :

$$\mathcal{L}(X)\Lambda = 0 \quad .$$

3°) The vector field X is a Poisson infinitesimal automorphism.

Proof. Equivalence of properties 1° and 2° is easy. In order to prove the equivalence of these two properties with property 3°, we remark that if φ is the integral flow of X, we have for all $t_o \in \mathbb{R}$:

$$\begin{aligned}\frac{d}{dt}\varphi^\star_{-t}\{\varphi^\star_t f , \varphi^\star_t g\}\Big|_{t=t_o} &= -\varphi^\star_{-t_o}(X.\{\varphi^\star_{t_o} f , \varphi^\star_{t_o} g\}) \\ &\quad + \varphi^\star_{-t_o}(\{X.(\varphi^\star_{t_o} f) , \varphi^\star_{t_o} g\}) \\ &\quad + \varphi^\star_{-t_o}(\{\varphi^\star_{t_o} f , X.(\varphi^\star_{t_o} g)\}) \quad .\end{aligned}$$

It is then easy to see that properties 2° and 3° are equivalent.

The last proposition shows in particular that the set of infinitesimal automorphisms of the Poisson manifold M, is a Lie subalgebra of the Lie algebra of differentiable vector fields on M.

4.4. Example

Let $f \in C^\infty(M , \mathbb{R})$ be a function on the Poisson manifold M. The associated Hamiltonian vector field #df (definition 2.6) is a Poisson infinitesimal automorphism of M. We have indeed, for all g and $h \in C^\infty(M , \mathbb{R})$:

$$\begin{aligned}\#df.\{g,h\} &= \{f , \{g,h\}\} \\ &= \{\{f,g\} , h\} + \{g , \{f,h\}\} \\ &= \{\#df.g , h\} + \{g , \#df.h\} \quad .\end{aligned}$$

The following definition generalizes the definitions of locally and globally Hamiltonian vector fields (Abraham and Marsden, 1978), which are well known for a symplectic manifold. We will see that, on a Poisson manifold, locally Hamiltonian vector fields are infinitesimal automorphisms of the Poisson structure; but infinitesimal Poisson automorphisms may exist, which are not locally Hamiltonian vector fields.

4.5. Definitions

1°) A differential p-form η on the Poisson manifold M is said $\mathfrak{F}$-closed if, for any family $(f_1, \dots f_{p+1})$ of p+1 differentiable functions on M, one has :

$$d\eta(\#df_1, \dots \#df_{p+1}) = 0 \quad .$$

2°) A vector field X on the Poisson manifold M is said locally Hamiltonian if there exists an $\mathfrak{F}$-closed Pfaff form α on M, such that :

$$X = \#\alpha \quad .$$

Any closed 1-form (and, therefore, the differential df of any differentiable function f on M) is $\mathfrak{F}$-closed. Hence a Hamiltonian vector field #df on M, is locally Hamiltonian.

4.6. Proposition

Let α be a Pfaff form on the Poisson manifold M. The vector field $\#\alpha$ is an infinitesimal Poisson automorphism if and only if α is $\mathfrak{F}$-closed, that means, if and only if $\#\alpha$ is locally Hamiltonian.

Proof. Let f and g be two elements of $C^\infty(M,\mathbb{R})$. We have :

$$\begin{aligned}
\#\alpha.\{f,g\} &= - \langle\alpha, \#d\{f,g\}\rangle \\
&= - \langle\alpha, [\#df , \#dg]\rangle \quad ; \\
\{\#\alpha.f , g\} &= \{\Lambda(\alpha,df) , g\} \\
&= - \{\langle\alpha , \#df\rangle , g\} \\
&= \#dg.(\langle\alpha , \#df\rangle) \quad ; \\
\{f , \#\alpha.g\} &= - \#df.(\langle\alpha , \#dg\rangle) \quad .
\end{aligned}$$

We obtain:

$$\#\alpha.\{f,g\} - \{\#\alpha.f , g\} - \{f , \#\alpha.g\} = d\alpha(\#df , \#dg) \quad ,$$

and the result follows from this equality .

4.7. Proposition

Let X and Y be two vector fields on the Poisson manifold M.

1°) If X is an infinitesimal Poisson automorphism and Y a locally Hamiltonian vector field, the bracket $[X,Y]$ is locally Hamiltonian; more precisely, if β is an $\mathcal{F}$-closed Pfaff form such that :

$$Y = \#\beta \quad ,$$

one has :

$$[X,Y] = \#(\mathcal{L}(X)\beta) \ .$$

2°) If X and Y are both locally Hamiltonian, the bracket $[X,Y]$ is a Hamiltonian vector field; more precisely, if α and β are two $\mathcal{F}$-closed Pfaff forms such that :

$$X = \#\alpha \quad , \quad Y = \#\beta \quad ,$$

one has :

$$[X,Y] = \#d(i(X)\beta) = -\#d(i(Y)\alpha) \ .$$

Proof. Let β be an $\mathcal{F}$-closed Pfaff form such that :

$$Y = \#\beta \ .$$

For any function $f \in C^\infty(M,\mathbb{R})$, we have :

$$\begin{aligned}[X,Y].f &= X.(Y.f) - Y.(X.f) \\ &= X.(\Lambda(\beta,df)) - \Lambda(\beta \ , d(X.f)) \\ &= \Lambda(\mathcal{L}(X)\beta \ , df)\end{aligned}$$

because the Lie derivative $\mathcal{L}(X)\Lambda$ of the Poisson tensor field Λ vanishes (proposition 4.3). Therefore we have, for any function $f \in C^\infty(M,\mathbb{R})$:

$$[X,Y].f = \#(\mathcal{L}(X)\beta).f \quad ,$$

and this shows that :

$$[X,Y] = \#\mathcal{L}(X)\beta \quad .$$

But as X and Y are infinitesimal Poisson automorphisms, their bracket $[X,Y]$ is also an infinitesimal Poisson automorphism; by proposition 4.6, we see that $\mathcal{L}(X)\beta$ is $\mathcal{F}$-closed, or that $[X,Y]$ is locally Hamiltonian. This completes the proof of 1°.

Under the hypotheses of 2°, by using the formula :

$$\mathcal{L}(X)\beta = d\, i(X)\beta + i(X)\, d\beta \quad ,$$

we obtain :

$$[X,Y] = \#d(i(X)\beta) + \#(i(X)\, d\beta) \quad .$$

But for any function $f \in C^\infty(M,\mathbb{R})$, we have :

$$\begin{aligned} \#(i(X)\, d\beta).f &= \Lambda(i(X)\, d\beta\ ,\ df) \\ &= d\beta(\#df\ ,\ X) \\ &= 0 \quad . \end{aligned}$$

The last equality is due to the facts that β is $\mathcal{F}$-closed, and that, X being locally Hamiltonian, for any point $x \in M$, there exists a differentiable function h on M such that :

$$X(x) = \#dh(x) \quad .$$

Therefore we have :

$$[X,Y] = \#d(i(X)\beta) \quad ,$$

which completes the proof of 2°.

The reader is referred to the paper of Lichnerowicz (1977) for a much more thorough study of the various Lie algebras of vector fields associated with a Poisson manifold, of their derivations, ideals and deformations. Results indicated above are, for their main part, adapted from the corresponding results established by Lichnerowicz in the particular case when the rank of the Poisson tensor field Λ is constant.

5. FUNCTIONS IN INVOLUTION ON A POISSON MANIFOLD

5.1. Definition

Let M be a Poisson manifold. Two functions f and $g \in C^\infty(M,\mathbb{R})$ are said in involution when :

$$\{f,g\} = 0 \quad .$$

The corresponding definition is well known for functions defined on a symplectic manifold. The importance of this concept is related to the classical Liouville theorem (Arnold, 1974, Arnold and Avez, 1967) about completely integrable Hamiltonian systems.

Following the work of Lax (1968) about isospectral deformations, several recent works were devoted to completely integrable Hamiltonian systems : see the papers referred to in the introduction, paragraph 1.5. In these works appear theorems which give conditions under which functions are in involution (see in particular the paper of T. Ratiu, 1980). Some of these theorems may be put under a simpler and more general form, when the concept of Poisson manifold is used. This is the case for the Adler-Kostant-Symes theorem, which may be formulated as follows.

5.2. Theorem (M. Selmi, 1982, and the author)

Let M and N be two Poisson manifolds, and $\varphi : M \to N$ a Poisson morphism, which is also a surjective submersion. Let $s : U \to M$ be a section of φ, that means, a differentiable map from an open subset U of N, into M, such that $\varphi \circ s$ is the identity map of U. We assume that the submanifold s(U) of M satisfies the following property:

Property P : for any pair (h,k) of differentiable functions on M, whose restrictions to s(U) are constant, the Poisson bracket $\{h,k\}$ vanishes on s(U).

Then if f and $g \in C^\infty(M,\mathbb{R})$ are two invariants of the Poisson structure of M (definition 3.1), the two functions $f \circ s$ and $g \circ s$, defined on the open subset U of N, are in involution:

$$\{f \circ s \, , \, g \circ s\} = 0 \quad .$$

Proof. We set :

$$U_1 = s(U) \quad .$$

Let x be a point of U, and $y = s(x)$ the corresponding point of U_1. We have the direct sum decomposition :

$$T_yM = T_yU_1 \oplus \ker T_y\varphi \quad .$$

By duality, we deduce the direct sum decomposition of the cotangent space $T^\star_yM$:

$$T^\star_yM = (\ker T_y\varphi)^\circ \oplus (T_yU_1)^\circ \quad ,$$

where $(\ker T_y\varphi)^\circ$ and $(T_yU_1)^\circ$ are the annihilators, respectively, of $\ker T_y\varphi$ and of T_yU_1 (that means, the vector subspaces of the cotangent space $T^\star_yM$, made of linear forms on T_yM which vanish, respectively, on $\ker T_y\varphi$ and on T_yU_1). The two subspaces $(\ker T_y\varphi)^\circ$ and $(T_yU_1)^\circ$ may be identified, respectively, with the dual spaces $T^\star_yU_1$ of T_yU_1, and $(\ker T_y\varphi)^\star$ of $\ker T_y\varphi$. We note :

$$\pi_1 : T^\star_yM \to (\ker T_y\varphi)^\circ \equiv T^\star_yU_1 \quad ,$$
$$\pi_2 : T^\star_yM \to (T_yU_1)^\circ \equiv (\ker T_y\varphi)^\star$$

the two projections defined by this direct sum decomposition. Let Λ_M be the Poisson tensor field on M. We have :

$$\{f \circ s \, , \, g \circ s\}(x) = \Lambda_M(\pi_1(df(y)) \, , \, \pi_1(dg(y)) \quad ,$$

because $\varphi : M \to N$ is a Poisson morphism. We may write :

$$\begin{aligned}\{f\circ s \, , \, g\circ s\}(x) &= \Lambda_M(df(y) - \pi_2(df(y)) \, , \, dg(y) - \pi_2(dg(y)))\\ &= \langle dg(y) - \pi_2(dg(y)) \, , \, \#df(y)\rangle\\ &\quad + \langle \pi_2(df(y)) \, , \, \#dg(y)\rangle\\ &\quad + \Lambda_M(\pi_2(df(y)) \, , \, \pi_2(dg(y))) \quad .\end{aligned}$$

But the first two terms of this last expression vanish, because as f and g are Poisson invariants, $\#df(y)$ and $\#dg(y)$ are equal to zero. On the other hand, $\pi_2(df(y))$ and $\pi_2(dg(y))$ belong to $(T_yU_1)^\circ$; there exist two functions h and k, constant on U_1, such that :

$$dh(y) = \pi_2(df(y)) \quad ; \quad dk(y) = \pi_2(dg(y)) \quad ,$$

and we have, by Property P :

$$\Lambda_M(\pi_2(df(y)) \, , \, \pi_2(dg(y))) = \{h,k\}(y) = 0 \quad .$$

Finally we have :

$$\{f \circ s \, , \, g \circ s\}(x) = 0 \quad .$$

5.3. Remark

When M is a symplectic manifold, property P of theorem 5.2 means that $s(U)$ is a coisotropic submanifold of M. When M is a Poisson manifold, it seems that submanifolds of M which verify Property P play a part very similar to that of coisotropic submanifolds of a symplectic manifold.

5.4. Application

We give here the usual form of the Adler-Kostant-Symes theorem and we will show how it can be deduced from theorem 5.2.

Let $\mathfrak{g}$ be a real, finite dimensional Lie algebra, $\mathfrak{h}$ and $\mathfrak{k}$ two Lie subalgebras of $\mathfrak{g}$ such that we have the direct sum vector space decomposition :

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \quad .$$

We have, for the dual space $\mathfrak{g}^\star$, the corresponding direct sum decomposition :

$$\mathfrak{g}^\star = \mathfrak{k}^\circ \oplus \mathfrak{h}^\circ \quad ,$$

where :

$$\mathfrak{k}^\circ \equiv \mathfrak{h}^\star \quad ; \quad \mathfrak{h}^\circ \equiv \mathfrak{k}^\star \quad .$$

On the spaces $\mathfrak{g}^\star$ and $\mathfrak{h}^\star$, we consider the Kirillov-Kostant-Souriau Poisson structures. Let $\lambda \in \mathfrak{g}^\star$ be such that :

$$(\star) \qquad \langle \lambda \, , \, [\mathfrak{h},\mathfrak{h}] \rangle = 0 \quad ;$$

$$(\star\star) \qquad \langle\lambda , [\mathfrak{k},\mathfrak{k}]\rangle = 0 \quad .$$

Let f and g be two functions on $\mathcal{G}^\star$, which are invariants of its Poisson structure. We note f_λ and g_λ the two functions defined on $\mathcal{G}^\star$ by :

$$f_\lambda(x) = f(x + \lambda) \quad , \quad (x \in \mathcal{G}^\star) \quad ;$$

$$g_\lambda(x) = g(x + \lambda) \quad , \quad (x \in \mathcal{G}^\star) \quad .$$

We note $i : \mathfrak{h}^\star \equiv \mathfrak{k}^0 \to \mathcal{G}^\star$ the canonical injection.

Then the two functions $f_\lambda \circ i$ and $g_\lambda \circ i$, defined on the Poisson manifold $\mathfrak{h}^\star$, are in involution :

$$\{f_\lambda \circ i \ , \ g_\lambda \circ i\} = 0 \quad .$$

In order to deduce this result from theorem 5.2, we take :

$$M = \mathcal{G}^\star \quad ; \quad N = \mathfrak{h}^\star \equiv \mathfrak{k}^0 \quad .$$

We define $\varphi : M \to N$, and $s : N \to M$, by :

$$\varphi(y) = \pi_1(y - \lambda) \quad , \quad (y \in \mathcal{G}^\star) \quad ,$$

$$s(x) = x + \lambda \quad , \quad (x \in \mathfrak{k}^0) \quad ,$$

where $\pi_1 : \mathcal{G}^\star = \mathfrak{k}^0 \oplus \mathfrak{h}^0 \to \mathfrak{k}^0$ is the first projection.

Using the property $(\star)$ above, we can check that φ is a Poisson morphism. Similarly, using the property $(\star\star)$, we see that the property P of theorem 5.2 is satisfied by the affine submanifold $s(\mathfrak{k}^0)$ of the Poisson manifold $\mathcal{G}^\star$. We can apply theorem 5.2, and we obtain the result indicated above.

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