Structure des Systèmes Dynamiques Jean-Marie Souriau's Book 50th Birthday

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Abstract. Jean-Marie Souriau's book "Structure des systèmes dynamiques", published in 1970, republished recently by Gabay, translated in English and published under the title "Structure of Dynamical Systems, a Symplectic View of Physics", is a work with an exceptional wealth which, fifty years after its publication, is still topical. In this paper, we give a brief description of its content and we intend to highlight the ideas that to us, are the most creative and promising.

1 A Few Introductory Words

Graduated of the Ecole Normale Supérieure in 1942, Jean-Marie Souriau obtains the aggregation of mathematics in 1945, ranked second. After a brief passage at the CNRS, he joins as an engineer the ONERA recently created and prepares his Ph.D thesis on the airplane stability. Defended in 1952, this work was used for the design of several airplanes, in particular the Concorde. Next he obtains a position as Professor at the Tunis Faculty in 1952 and at Aix-Marseille University in 1964, where he published "Structure des Systèmes Dynamique". For more details on Jean-Marie Souriau's life and personality from his friends and persons who have known him very well, you can read the paper paying hommage to him in [9].

Jean-Marie Souriau's book "Structure des systèmes dynamiques" [6], was published in 1970 in a book collection for students in the first year of master's degree in Mathematics, directed in fact to mathematicians, beginners or experienced, wishing to know the applications of mathematics to physical sciences, and to physicists concerned with knowing certain mathematical tools useful for their researches. The author was very aware of this since, in his Introduction, he gives reading recommendations adapted to both reader categories. The book is illustrated by many figures which mostly are very meaningful schematic representations of the geometric constructions used by the author. The notations are often very non standard but they are totally consistent.

The original edition is out of print, although it is possible to find copies of second hand. Nevertheless, it was republished recently by Gabay. It was also translated in English and published under the title "Structure of Dynamical

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Systems, a Symplectic View of Physics" [7]. The traduction was supervised by Richard Cushman and Gijs Tuynman who have known very well the author. It is also possible to download a scanned version from Jean-Marie's Souriau official website [8], and a lot of other interesting thinks.

2 Introduction

The book begins by a large introduction of 20 pages in which Souriau highlights the guiding ideas. He considers the classical analytical mechanics, that originated in the book by Jean-Louis Lagrange *Mécanique analytique* [3] and remains an essential ingredient of the current physical theories, is not outdated although certain used concepts have become so because they have not the required covariance, in other words because they are in contradiction with Galilean relativity. He wishes to show in his book that a better consideration of Lagrange's thought allows giving to this theory a form compatible with the more recent discoveries of the physical sciences.

3 Chapter I: Differential Geometry

In the first chapter, Souriau presents in less than 70 pages numerous tricky notions from differential manifolds to Lie groups and calculus of variation. The author presents the concept of differential manifold in a rather original way, bypassing the one of topological space, undoubtedly to make this notion easily accessible to beginning students. The differential manifolds are not supposed Hausdorff. Examples of non Hausdorff manifolds such as the space of motions of certain mechanical systems will be indeed encountered in Sect. 5.

Note author's language particularity that could cause confusion: he calls embedding what most of geometers call injective immersion. This was corrected in the English version. This choice is very reasonable because injective immersions are much more frequently encountered than embeddings. For example, orbits of a Lie group action, as well as leaves of a foliation, always are immersed in the manifold in which they are contained, and much more rarely embedded.

4 Chapter II: Symplectic Geometry

The second chapter is also essentially mathematical but shorter than the previous one (about 45 pages). It starts with algebraic notions concerning the skew-symmetric forms, then 2-forms and denoted σ. If it is non degenerate, it is called symplectic, otherwise presymplectic.

Next, Souriau defines the symplectic and presymplectic manifolds, and studies their properties. He shows that under certain conditions the quotient of a presymplectic manifold by its kernel is a symplectic manifold, result that he will use latter on to define the space of motions of a dynamical system that plays a central role in his theory. The elements of the quotient are leaves of a characteristic foliation.

Let u be a differentiable function defined on a symplectic manifold. Its differential du is a 1-form. If the form σ is symplectic, there exists a unique vector associated to du that he denotes $grad\, u$ and call symplectic gradient, such that:

$$-du \equiv \sigma(grad\, u)$$

The flow of this vector field let the symplectic form invariant. Let us remark in passing the non standard notation for the interior product $\sigma(grad\, u)$ of the vector $grad\, u$ and the form σ.

Souriau calls dynamical group (that other authors call symplectic group) of a symplectic or presymplectic manifold V a Lie group G acting on it by canonical transformations. He calls moment of the dynamical group G a differentiable map ψ from a point x of V onto an element μ of the vector space $\mathcal{G}^*$, dual of the Lie algebra $\mathcal{G}$ of G, such that for every generator $Z \in \mathcal{G}$ its infinitesimal action is the symplectic gradient of the function linking, at every $x \in V$ the real $\mu \cdot Z$:

$$\sigma(Z_V(x)) \equiv -d\,[\mu \cdot Z]$$

In the terminology used by most of geometers, this infinitesimal generator is the Hamiltonian vector space, the corresponding Hamiltonian being this function of value $\mu \cdot Z$.

The author gives several examples of dynamical groups, indicates suffisant conditions for a dynamical group admitting a moment and studies its properties. This deals him to propose a generalization of Noether's theorem.

We arrive to one of the most original Souriau's theories, the symplectic cohomology of Lie Groups.

In one hand, we let the element a act on x, next we take the value of the moment map. On the other hand, we take the value μ of the moment map ψ, next we apply the coadjoint representation of the group for the element a of G. Souriau proved that the difference between the two previous results does not depends on the current point x on the manifold, then only on a. This defines a map:

$$\theta(a) \equiv \psi(\underline{a}_V(x)) - \underline{a}_{\mathcal{G}^*}(\psi(x))$$

from the group to the dual of its Lie algebra. It verifies an identity that means that the group acts on the moment space by affine representation with linear part the coadjoint action $\underline{a}_{\mathcal{G}^*}$ and translation part $\theta(a)$:

$$\theta(a \times b) \equiv \theta(a) + \underline{a}_{\mathcal{G}^*}(\theta(b))$$

The differential of θ at the identity, denoted f, is a bilinear form on the dual and, even less obvious, it is skew-symmetric, then a 2-form and it verifies an identity that generalizes Jacobi's one:

$$f(Z)([Z', Z'']) + f(Z')([Z'', Z]) + f(Z'')([Z, Z']) = 0$$

The map θ measures the defect of symplectic cohomology of the dynamical group. It is called symplectic cocycle. It is possible to define also symplectic coboundaries, and passing to the quotient to define classes of symplectic cohomology.

We arrive to another fundamental result called now Kirillov-Kostant-Souriau theorem. In fact, it is not necessary to study case-by-case the action a dynamical group on every symplectic manifold. You can disregard the manifold and take interest on the action of the group onto the dual of its Lie algebra. Then, on each orbit U, there exists a canonical symplectic structure and the identity map of the orbit is a moment map. In other words, we may say that each point μ of the orbit U is its own moment.

5 Chapter III: Mechanics

The starting point is the following 2-form that Souriau attributes to Lagrange. The Lagrange form of a system of material points is the sum of the Lagrange forms of all points of the system, where m_j is the mass of particle j, $\boldsymbol{r}_j$ its position, $\boldsymbol{v}_j$ its velocity and $\boldsymbol{F}_j$ the resultant of the forces acting upon it:

$$\sigma = \sum_j \left(\left\langle m_j \delta \boldsymbol{v}_j - \boldsymbol{F}_j \delta t, \delta' \boldsymbol{r}_j - \boldsymbol{v}_j \delta' t \right\rangle - \left\langle m_j \delta' \boldsymbol{v}_j - \boldsymbol{F}_j \delta' t, \delta \boldsymbol{r}_j - \boldsymbol{v}_j \delta t \right\rangle \right)$$

that could written:

$$\sigma = \sum_j \left(m_j d\boldsymbol{v}_j - \boldsymbol{F}_j dt \right) \dot{\wedge} \left(d\boldsymbol{r}_j - \boldsymbol{v}_j dt \right)$$

where the symbol $\dot{\wedge}$ denotes the operator combining the dot product and the exterior product. You can observe here one of the peculiarities of Souriau's notations. He does not use the symbol $\wedge$ for the exterior product of differential forms, which has some advantages but also drawbacks. Then, the well-known equations of motion can be written in a very compact manner: the tangent vector to the trajectory in the evolution space of the positions, velocities and time belongs to the kernel of the presymplectic form.

Next, Souriau defines the vector $\boldsymbol{E}_j$ in terms of the force $\boldsymbol{F}_j$ occurring in Lagrange 2-forms and a vector $\boldsymbol{B}_j$ freely chosen:

$$\boldsymbol{E}_j \equiv \boldsymbol{F}_j + \boldsymbol{B}_j \times \boldsymbol{v}_j$$

This leads him to formulate what he calls Maxwell's principle in the following form: the vector fields $\boldsymbol{B}_j$ may be chosen in such a way that the Lagrange form is closed. Hence this condition determines these fields in a unique manner. Stating this principle, the author proves that:

1. the fields $\boldsymbol{E}_j$ and $\boldsymbol{B}_j$ must not depend on the velocities of the material points
2. $\boldsymbol{B}_j$ does not depends on the position of other particles than j
3. They verify a condition in which we recognize Maxwell reciprocity principle (in fact a version of the action and reaction principle for forces acting at distance)

4. And they verify two equations in which we recognize Faraday equation and the law of induction (or absence of magnetic monopoles). By contrast, Ampere and Gauss equations are relativistic effects which do not appear.

Souriau adopts Maxwell's principle as a new law of the mechanics (not only of the electromagnetism) and presents various examples: the N-body problem of Celestial mechanics, the gravitation, where $\boldsymbol{E}$ is the usual gravity force and $\boldsymbol{B} \times \boldsymbol{v}$ is Coriolis' force, and of course the electromagnetic field. Without denying the importance of the principle of least action or even the usefulness of these formalisms, the author declares these concepts seem to him less fundamental as the Maxwell principle.

One of the originally of Souriau's work is to emphasize the role played by the space of motions. Starting from the evolution space which is equipped with a presymplectic structure, the space of motions is a symplectic manifold obtained by quotient of the foliation. The space of motions of a dynamical system is not very often considered by modern authors, though it appeared as soon as 1808 in the works of Lagrange. This very natural concept has a nice mathematical property: the space of motions is always endowed with a smooth manifold structure.

One may wonder why the concept of space of motions is not used more by modern authors. Maybe it is because for some dynamical systems, the space of motions is a non-Hausdorff manifold. Another possible explanation is that some scientists are interested in the thorough description of particular motions of a system, rather than by the study of the set of all possible motions. By showing that many important results can be deduced from the symmetries of the space of motions of a system, the author proves that this reluctance is unfounded.

Next Souriau introduce Galilei group, the symmetry group of Mechanics, a Lie group of dimension 10. defined as the set of matrices :

$$a \equiv \begin{bmatrix} A & \boldsymbol{b} & \boldsymbol{c} \\ 0 & 1 & e \\ 0 & 0 & 1 \end{bmatrix},$$

where, with his notations, A is a rotation, $\boldsymbol{b}$ the Galilean boost, $\boldsymbol{c}$ a space translation ans e a clock change. When the system is isolated, the Galilei group acts onto the evolution space and the space of motions. On the evolution space, the moment of this action is a first integral of the motion. The author details its 10 components, which can be regrouped in three vectors $\boldsymbol{p}$, $\boldsymbol{l}$, $\boldsymbol{g}$ and a scalar E. He gives their physical meaning:

1. $\boldsymbol{p}$ is the total linear momentum;
2. $\boldsymbol{l}$ is the total angular momentum;
3. the equality $\boldsymbol{g} =$ Constant conveys the fact that the center of mass moves on a straight line at constant velocity;
4. the scalar E, defined modulo an additive constant, is the total energy.

By a calculation similar to Bargmann's one, Souriau proved that the symplectic cohomology of the Galilei group is of dimension 1. He considers the number

m that spots the class of cohomology of the action of the Galilei group on the space of motions as being the mass.

In the paragraph The principles of symplectic mechanics, the author first takes place in the frame of the classical mechanics, non relativistic. He is no longer limited to the systems of material points and he adopts the three following assertions as new axioms of the mechanics:

I. The space of the motions of a dynamical system is a connected symplectic manifold.
II. If several dynamical systems evolve independently, the manifold of motions of the composite system is the symplectic direct product of the spaces of motions of the component systems.
III. If a dynamical system is isolated, its manifold of motions admits the Galilei group as a dynamical group.

It is an extension of the principles generally admitted of the classical mechanics, which will allow to the author considering new dynamical systems having a physical interest.

In special relativity, the passage of a Lorentz frame to another one is made by the action of an element of the restricted Poincaré group. The transition to the Relativistic case consists in a simple change of the symmetry group in the third axioms:

III. If a dynamical system is isolated, its manifold of motions admits the restricted Poincaré group as a dynamical group.

In a long section, the author proposes a mechanistic description of elementary particles. In the framework of relativistic Mechanics, an isolated dynamical system is said to be elementary when the Poincaré group acts transitively on the space of its motions. The moment map of its action is then a symplectic diffeomorphism of this space onto a coadjoint orbit of the Poincaré group. For him, so defined elementary systems are mathematical models for the elementary particles of physicists. Besides the classical energy-momentum P, Souriau introduces the polarization 4-vector $W = *(M) \cdot P$, built from P and the moment M associated to the infinitesimal Lorentz transformation, through a linear map $*$ transforming an anti-hermitian operator into another anti-hermitian operator. As example let us consider a particle with spin. It is when P is timelike and when W (which, being orthogonal to P, is spacelike) is non-zero. It is characterized by two invariants, the mass m stemming from the energy-momentum and the spin s from the polarization.

The classification of the coadjoint orbits of Poincaré group provides us the table of elementary particles:

I. A particule with spin (previously described)
II. A particle without spin. It is when P is timelike and $W = 0$
III. A massless particle. It is when both P and W are non-zero and lightlike. The author defines three real numbers, the sign of the energy η, the helicity χ and the spin s

The Nonrelativistic particles are obtained by scaling of lengths and times, next considering the limit when the velocity approaches zero.

6 Chapter IV: Statistical Mechanics

It is devoted to Statistical Mechanics. It contains two sections. The first one is a very condensed course in Measure Theory and Integration, with some notions in Probability. The second one presents the principles of Statistical Mechanics in a very original way based on the Lie group theory. Souriau calls generalized Gibbs probability law any completely continuous probability law of density:

$$f(x) \equiv e^{-[z+Z(\Psi(x))]}$$

where z is a scalar, Z an infinitesimal generator living in the Lie algebra of the group and Ψ is the moment map such that the law is integrable. Then, he establishes the relation between the entropy s, the Planck potential z and the mean value M of the moment Ψ:

$$s = z + Z(M)$$

The author explains that the entropy of the system increases with time, so assuming that the natural equilibria of the gas are elements of the Gibbs set of the group of time translations is a very reasonable assumption. Each Gibbs state is determined by an element Z of the one-dimensional Lie algebra of this group, which is a way of measuring the gas temperature. Since the group of time translations is a subgroup, but not a normal subgroup, of the Galilei group, a dynamical system conservative in some inertial reference frame is not conservative in a different inertial reference frame. This important remark leads the author to introduce the new concept of covariant statistical mechanics by proposing the following principle:

When a dynamical system is invariant by the action of some Lie subgroup G' of the Galilei group, its natural equilibria are the elements of the Gibbs set of the action of G' .

He then discusses in greater detail several examples, among them a gas in a centrifuge (this relative motion is now a rotation around an axis at a constant angular velocity). He considers also a system made by particles with spin, and finds that the most probable orientation of the particles spin is parallel to the rotation axis.

7 Chapter V: A Method of Quantization

The theory of geometric quantization was developed in the early 70' independently by Souriau and Kostant [2] in order to bring into focus some vaguely perceived analogies between representation theory and quantum mechanics that had for a long time been part of the "flolklore" of modern physics. The author defines a quantum manifold as a smooth manifold Y endowed with a contact 1-form ϖ, a presymplectic structure for the exterior differential σ of ϖ, such that all integral curves are the orbits of an action on Y of the 1-dimensional torus. The set of these curves, in other words the quotient of Y by this action, is a symplectic manifold U, called by the author the base of the quantum manifold Y.

In practice, the Physicist only knows the space of motion U of a classical system and he has to address the following issue: is there a quantum manifold Y and a symplectomorphism from the base of Y onto the space of motion? This is the problem of the quantization. The author proves that the manifold of motions of a non-relativistic particle with spin is quantizable if and only if the spin of the particle is integer or half integer, when expressed with $\hbar$ as unit.

Isomorphisms of quantum manifolds are called by the author quantomorphisms. Any quantomorphism between two quantum manifolds projects onto a symplectomorphism between their bases. A group Γ of quantomorphisms of a quantum manifold Y projects onto a group of symplectomorphisms of its basis U, and its projection is a group homomorphism. Conversely, a group G of symplectomorphisms of the basis U is said to be liftable if there exists a group Γ of quantomorphisms of Y which projects onto it. He then discusses the quantization of a dynamical group of a quantizable symplectic manifold U, with the quantum manifold Y as quantization. He proves that when a dynamical group of U is quantizable, its symplectic cohomology is zero. He gives examples which prove that this necessary condition is not sufficient. A dynamical group of U which is liftable, but not quantizable, may have an extension which still is a dynamical group of U and is quantizable.

All this deep analysis on geometric basis allow to give solid foundations to the well known correspondance principle. Let u be a smooth function on the space of motions U, representing an observable. Then Souriau explains how to associate to any observable u defined on the base U an operator $\hat{u}$ on the Hilbert space $\mathcal{H}(Y)$ of the state vectors Ψ of the quantum manifold Y, defined by this relation :

$$\hat{u}(\Psi)(\xi) \equiv -i \underset{u}{\delta} [\Psi(\xi)]$$

where in the right hand member, the notation $\underset{u}{\delta}$ means the derivative of Ψ in the direction of the symplectic gradient of u Then Souriau proved that:

1. The operator $\hat{u}$ is hermitian,
2. The map from u to $\hat{u}$ is linear and injective,
3. For $u = 1$, $\hat{u}$ is the identity of the Hilbert space,

and he proved the correspondance between quantum commutators and Poisson brackets (originally proposed by Dirac).

8 Conclusions

In this brief presentation, we have just skimmed over the book content of exceptional depth. A more detailled presentation of the book will be published soon in [1].

When reading this book, we cannot fail to be impressed by the extent and the thoroughness of the author's knowledge, as well in Mathematics as in Mechanics or in Physics, and by the originality and the depth of his thoughts.

We would like also to bring our attention to the fact that Jean-Marie Souriau is the author of two other very remarkable books in French, both published in 1964, « Géométrie et Relativité" [4] and "Calcul linéaire" [5]. In that respect, we would like to recall the origin of the last book. From 1958, while Souriau is still engineer at ONERA, his taste for the teaching drives him on to create a free course (and free of charge) entitled "New methods of the Mathematical physics", as shown in this poster. It is a great success, the amphitheater, even able to contain 200 persons, is full and he has to teach twice. The program of the linear algebra gave rise to the book "Calcul Linéaire". "Géométrie et Relativité" and "Calcul linéaire", which too are very rich and original, deserve —as "Structure des systèmes dynamiques"— to be read, ... and read again.

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