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EXAMPLES OF GIBBS STATES OF MECHANICAL SYSTEMS WITH SYMMETRIES

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Abstract. In a previous paper, the notion of Gibbs state for the Hamiltonian action of a Lie group on a symplectic manifold was given, togeter with its applications in Statistical Mechanics, and the works in this field of the French mathematician and physicist Jean-Marie Souriau were presented. Using an adaptation of the cross product for pseudo-Euclidean three-dimensional vector spaces, we present in the present paper several examples of such Gibbs states, together with the associated thermodynamic functions, for various two-dimensional symplectic manifolds, including the pseudo-spheres, the Poincaré disk and the Poincaré half-plane.

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Contents

1. Introduction

Gibbs states for the Hamiltonian action of a Lie group on a symplectic manifold appeared for the first time, for the action of the group of rotations on the phase space of a mechanical system, in the book [4], published in 1902, in which the famous American scientist Josuah Willard Gibbs (1839–1903) laid the foundations of Statistical Mechanics. In several papers and in his book [13], the French mathematician and physicist Jean-Marie Souriau (1922–2012) studied these Gibbs states in full generality, for the Hamiltonian action of any Lie group on a symplectic manifold. In a previous paper [7], I discussed the importance of these Gibbs states in Statistical Mechanics and gave a detailed description of Souriaus's works in this field, with full proofs of all the stated results. In the present paper, examples of such Gibbs states are given for various two-dimensional symplectic manifolds, including the spheres, the pseudo-spheres, the Poincaré disk and the Poincaré half-plane. It uses the definitions and notations given in my previous paper, and begins by Section 2, in which the well known *cross product* of vectors in a three-dimensional oriented Euclidean vector space is extended to vectors in a three-dimensional oriented pseudo-Euclidean vector space. Remarkably enough, the cross-product of vectors in a three-dimensional oriented Euclidean vector space appeared for the first time in the book [3], privately written in 1881 for students in physics by Gibbs, one of the most important founders of Statistical Mechanics. By using this extension of the cross product, a remarkable isomorphism of the considered vector space onto the Lie algebra of its group of symmetries (which can be either $\mathrm{SO}(3)$ or $\mathrm{SO}(2,1)$ is obtained. It is used in Section 3 for te study of Gibbs states on various two-dimensional symplectic manifold. Examples of symplectic manifolds with symmetries on which no Gibbs state can exist are presented too.

2. Three-dimensional real oriented vector spaces with a scalar product

2.1. Admisible bases and symmetry groups

In what follows, ζ is a real integer whose value is either $+1$ or -1, and $\mathbf{F}$ is a three-dimensional real vector space endowed with a scalar product $\mathbf{F}\times\mathbf{F}\to\mathbb{R}$, denoted by $(\mathbf{v},\mathbf{w})\mapsto\mathbf{v}\cdot\mathbf{w}$, with $\mathbf{v}$ and $\mathbf{w}\in\mathbf{F}$, whose signature is $(+,+,+)$ when $\zeta=1$ and $(+,+,-)$ when $\zeta=-1$. This scalar product is Euclidean when $\zeta=1$ and pseudo-Euclidean when $\zeta=-1$. A basis $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ of $\mathbf{F}$ is said to be *orthonormal* when

$$\mathbf{e}_x\cdot\mathbf{e}_x=\mathbf{e}_y\cdot\mathbf{e}_y=1\,,\ \mathbf{e}_z\cdot\mathbf{e}_z=\zeta\,,\ \mathbf{e}_x\cdot\mathbf{e}_y=\mathbf{e}_y\cdot\mathbf{e}_z=\mathbf{e}_z\cdot\mathbf{e}_x=0\,.$$

When $\zeta=-1$, the vector space $\mathbf{F}$ is called a *three-dimensional Minkowski vector space*. A non-zero element $\mathbf{v}\in\mathbf{F}$ is said to be *space-like* when $\mathbf{v}\cdot\mathbf{v}>0$, *time-like* when $\mathbf{v}\cdot\mathbf{v}<0$ and *light-like* when $\mathbf{v}\cdot\mathbf{v}=0$. The subset of $\mathbf{F}$ made of non-zero time-like or light-like elements has two connected components. A *temporal orientation* of $\mathbf{F}$ is the choice of one of these two connected components, whose elements are said to be *directed towards the future*. Elements of the other connected component are said to be *directed towards the past*.

Both when $\zeta=1$ and when $\zeta=-1$, we will assume in what follows that an orientation of $\mathbf{F}$ in the usual sense is chosen, and when $\zeta=-1$, we will assume that a temporal orientation of $\mathbf{F}$ is chosen too. The orthonormal bases of $\mathbf{F}$ used will always be chosen positively oriented and, when $\zeta=-1$, their third element $\mathbf{e}_z$ will be chosen time-like and directed towards the future. Such bases of $\mathbf{F}$ will be called *admissible bases*.

We denote by G the subgroup of $\mathrm{GL}(\mathbf{F})$ made of linear automorphisms g of $\mathbf{F}$ which transform any admissible basis $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ of $\mathbf{F}$ into an admissible basis $\bigl(g(\mathbf{e}_x),g(\mathbf{e}_y),g(\mathbf{e}_z)\bigr)$. Elements g of G preserve the scalar product in $\mathbf{F}$, *i.e.*, they are such that, for any pair $(\mathbf{v},\mathbf{w})\in\mathbf{F}\times\mathbf{F}$,

$$g(\mathbf{v})\cdot g(\mathbf{w})=\mathbf{v}\cdot\mathbf{w}\,.$$

Moreover, they preserve the orientation of $\mathbf{F}$ and, when $\zeta=-1$, its temporal orientation. The subgroup G of $\mathrm{GL}(\mathbf{F})$ is the group of symmetries of $\mathbf{F}$, endowed with its scalar product, its orientation and, when $\zeta=-1$, its temporal orientation. It is a connected Lie group isomorphic to the rotation group $\mathrm{SO}(3)$ when $\zeta=1$, and to the restricted three-dimensional Lorentz group $\mathrm{SO}(2,1)$ when $\zeta=-1$. Its Lie algebra, which will be denoted by $\mathfrak{g}$, is therefore isomorphic to $\mathfrak{so}(3)$ when $\zeta=1$, and to $\mathfrak{so}(2,1)$ when $\zeta=-1$.

Some useful properties of the vector space $\mathbf{F}$, of its symmetry group G and of the Lie algebra $\mathfrak{g}$ are recalled below. The interested reader will find their detailed proofs in [6] or, for most of them, in the very nice book [9]. The very nice results about parametrizations of rotations and their vector decompositions presented in [1, 8] are closely related to the properties of Euclidean oriented three-dimensional vector spaces discussed below. It would be interesting to look at their possible extension when the three-dimensional oriented space considered is pseudo-Euclidean.

2.2. A remarkable Lie algebras isomorphism

Let $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an admissible basis of $\mathbf{F}$, in the sense given in Subsection 2.1. For any triple $(a, b, c) \in \mathbb{R}^3$, let $j(a\mathbf{e}_x + b\mathbf{e}_\mathbf{y} + c\mathbf{e}_z)$ be the linear endomorphism of $\mathbf{F}$ whose matrix, in the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, is

$$\text{matrix of } j(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z) = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -\zeta b & \zeta a & 0 \end{pmatrix} .$$

The map j does not depend on the admissible base of $\mathbf{F}$ used for its definition. This property follows from the fact that j can be expressed in terms of the Hodge star operator, as explained below in Subsection 2.3. It is linear and injective, and its image is the Lie algebra $\mathfrak{g}$, considered as a vector subspace of the vector space $\mathcal{L}(\mathbf{F}, \mathbf{F})$ of linear endomorphisms of $\mathbf{F}$. There exists a unique bilinear and skew-symmetric map, defined on $\mathbf{F} \times \mathbf{F}$ and with values in $\mathbf{F}$, denoted by $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \dot{\times} \mathbf{w}$, such that, for all $\mathbf{v}$ and $\mathbf{w} \in \mathbf{F}$,

$$j(\mathbf{v} \dot{\times} \mathbf{w}) = \big[j(\mathbf{v}), j(\mathbf{w})\big] = j(\mathbf{v}) \circ j(\mathbf{w}) - j(\mathbf{w}) \circ j(\mathbf{v}) .$$

The bilinear map $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \dot{\times} \mathbf{w}$ will be called the *cross product* on $\mathbf{F}$. The map $j : \mathbf{F} \to \mathfrak{g}$ is a Lie algebras isomorphism of $\mathbf{F}$ (endowed with the cross product as composition law) onto the Lie algebra $\mathfrak{g}$, whose composition law is the commutator of endomorphisms. Its transpose $j^T : \mathfrak{g}^* \to \mathbf{F}^*$, defined by the equality

$$\langle j^T(\xi), \mathbf{v} \rangle = \langle \xi, j(\mathbf{v}) \rangle , \quad \xi \in \mathfrak{g}^* , \ \mathbf{v} \in \mathbf{F} ,$$

is therefore an isomorphism of the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ onto the dual vector space $\mathbf{F}^*$ of $\mathbf{F}$.

When $\zeta = 1$, the cross product is the well known *cross product* $(\mathbf{v}, \mathbf{w}) \mapsto \mathbf{v} \times \mathbf{w}$ on the Euclidean oriented three-dimensional vector space $\mathbf{F}$, and the map j is the isomorphism of $\mathbf{F}$ onto the Lie algebra $\mathfrak{g} \equiv \mathfrak{so}(3)$ of its Lie group of symmetries

$G \equiv \mathrm{SO}(3)$, very often used in Mechanics (see for example [13]). These remarkable properties of oriented Euclidean three-dimensional vector spaces therefore still hold for oriented pseudo-Euclidean three-dimensional vector spaces, the usual cross product being replaced with the cross product defined above.

Both when $\zeta = 1$ and when $\zeta = -1$, for all $g \in G$, $\mathbf{v}$ and $\mathbf{w} \in \mathbf{F}$,

$$j\bigl(g(\mathbf{v})\bigr) = \mathrm{Ad}_g\bigl(j(\mathbf{v})\bigr)\,, \quad g(\mathbf{v})\dot{\times} g(\mathbf{w}) = g(\mathbf{v}\dot{\times}\mathbf{w})\,.$$

The first above equality expresses the fact that the map j is equivariant with respect to the natural action of G on $\mathbf{F}$ and its adjoint action on its Lie algebra $\mathfrak{g}$. The second expresses the fact that the action of the group of symmetries G preserves the cross product.

We denote by scal the linear map defined by the equality

$$\bigl\langle \mathrm{scal}(\mathbf{u}), \mathbf{v}\bigr\rangle = \mathbf{u}\cdot\mathbf{v} \quad \text{for all } \mathbf{u} \text{ and } \mathbf{v} \in \mathbf{F}\,,$$

where, in the left hand side, $\bigl\langle \mathrm{scal}(\mathbf{u}), \mathbf{v}\bigr\rangle$ denotes the pairing by duality of $\mathrm{scal}(\mathbf{u}) \in \mathbf{F}^*$ with $\mathbf{v} \in \mathbf{F}$. The map scal is an isomorphism of $\mathbf{F}$ onto its dual vector space $\mathbf{F}^*$, which satisfies, for any admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$,

$$\mathrm{scal}(\mathbf{e}_x) = \boldsymbol{\varepsilon}_x\,, \quad \mathrm{scal}(\mathbf{e}_y) = \boldsymbol{\varepsilon}_y\,, \quad \mathrm{scal}(\mathbf{e}_z) = \zeta\boldsymbol{\varepsilon}_z\,,$$

where $(\boldsymbol{\varepsilon}_x, \boldsymbol{\varepsilon}_y, \boldsymbol{\varepsilon}_z)$ is the basis of $\mathbf{F}^*$ dual of the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$.

The isomorphism $\mathrm{scal} : \mathbf{F} \to \mathbf{F}^*$ satisfies, for all $g \in G$ and $\mathbf{v} \in \mathbf{F}$,

$$\mathrm{scal}(g(\mathbf{v})) = (g^{-1})^T(\mathrm{scal}\,\mathbf{v})\,,$$

where $(g^{-1})^T : \mathbf{F}^* \to \mathbf{F}^*$ is the linear automorphism of $\mathbf{F}^*$ transpose of the linear automorphism g^{-1} of $\mathbf{F}$. This equality expresses the fact that scal is equivariant with respect to the natural action of G on $\mathbf{F}$ and its contragredient action on the left on $\mathbf{F}^*$, $(g, \eta) \mapsto (g^{-1})^T(\eta)$, with $g \in G$, $\eta \in \mathbf{F}^*$.

Therefore the map $(j^{-1})^T \circ \mathrm{scal} : \mathbf{F} \to \mathfrak{g}^*$ is a linear isomorphism which satisfies, for all $g \in G$ and $\mathbf{v} \in \mathbf{F}$,

$$(j^{-1})^T \circ \mathrm{scal}\bigl(g(\mathbf{v})\bigr) = \mathrm{Ad}^*_{g^{-1}}\bigl((j^{-1})^T \circ \mathrm{scal}(\mathbf{v})\bigr)\,,$$

which expresses the fact that the isomorphism $(j^{-1})^T \circ \mathrm{scal}$ is equivariant with respect to the natural action of G on $\mathbf{F}$ and its coadjoint action on the left on $\mathfrak{g}^*$, $(g, \xi) \mapsto \mathrm{Ad}^*_{g^{-1}}\xi$, with $g \in G$, $\xi \in \mathfrak{g}$.

In what follows the vector space $\mathbf{F}$ will be identified either with the Lie algebra $\mathfrak{g}$ by means of the isomorphism j, or with the dual vector space $\mathfrak{g}^*$ by means of the isomorphism $(j^{-1})^T \circ \mathrm{scal}$. We will write simply $\mathbf{F} \equiv \mathfrak{g}$ when $\mathbf{F}$ is identified with $\mathfrak{g}$ and $\mathbf{F} \equiv \mathfrak{g}^*$ when it is identified with $\mathfrak{g}^*$, without writing explicitly the isomorphism used for this identification. The natural action of G on F will therefore be identified with its adjoint action on $\mathfrak{g}$ when $\mathbf{F} \equiv \mathfrak{g}$ and with its coadjoint action on the left on $\mathfrak{g}^*$ when $\mathbf{F} \equiv \mathfrak{g}^*$.

2.3. Expression of the map j in terms of the Hodge star operator

For any oriented n-dimensional real vector space endowed with a nondegenerate scalar product with any signature, the *Hodge star operator*, introduced by the British mathematician W. V. D. Hodge (1903–1975), is a linear automorphism of the vector space $\bigwedge V = \oplus_{k=0}^{n} \bigwedge^k V$ wich, for each integer k satisfying $0 \leq k \leq n$, maps $\bigwedge^k V$ onto $\bigwedge^{n-k} V$, with, by convention, $\bigwedge^0 V = \mathbb{R}$ (see for example [14] or [2], page 281). For the three-dimensional vector space $\mathbf{F}$ considered here, the Hodge star operator satisfies, for any admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$, the following equalities:

$$
\begin{aligned}
*(1) &= \mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z\,, &&\text{and conversely } *(\mathbf{e}_x \wedge \mathbf{e}_y \wedge \mathbf{e}_z) = \zeta\,,\\
*(\mathbf{e}_x) &= \mathbf{e}_y \wedge \mathbf{e}_z\,, &&\text{and conversely } *(\mathbf{e}_y \wedge \mathbf{e}_z) = \zeta \mathbf{e}_x\,,\\
*(\mathbf{e}_y) &= \mathbf{e}_z \wedge \mathbf{e}_x\,, &&\text{and conversely } *(\mathbf{e}_z \wedge \mathbf{e}_x) = \zeta \mathbf{e}_y\,,\\
*(\mathbf{e}_z) &= \zeta \mathbf{e}_x \wedge \mathbf{e}_y\,, &&\text{and conversely } *(\mathbf{e}_x \wedge \mathbf{e}_y) = \mathbf{e}_z\,.
\end{aligned}
$$

By using these formulae, one easily can check that the isomorphism $j : \mathbf{F} \to \mathfrak{g}$ is expressed in terms of the Hodge star operator as follows. For any triple $(a, b, c) \in \mathbb{R}^3$,

$$j(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z) = *(\zeta a\mathbf{e}_x + \zeta b\mathbf{e}_y + c\mathbf{e}_z)\,.$$

This result immediatly implies that the isomorphism j does not depend on the choice of the admissible basis used for its definition. When I first introduced j when $\zeta = -1$ in [6], I was not aware of its expression in terms of the Hodge star operator. With a better choice of conventions for the definition of j, its expression in terms of the Hodge star operator could be made more natural.

2.4. Metric, Lie algebra and Lie-Poisson structures of F

Since, as explained at the end of Subsection 2.2, we have both $\mathbf{F} \equiv \mathfrak{g}$ and $\mathbf{F} \equiv \mathfrak{g}^*$, the vector space $\mathbf{F}$ is endowed with a Lie algebra structure for which its identification with $\mathfrak{g}$ is a Lie algebras isomorphism, and with a Lie-Poisson structure for which its identification with $\mathfrak{g}^*$ is a Poisson diffeomorphism. Let $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an admisssible basis of $\mathbf{F}$, and let x, y and z be the coordinate functions on $\mathbf{F}$ in this admissible basis.

As seen in Subsection 2.2, the composition law of the Lie algebra structure of $\mathbf{F}$ is the cross product $(\mathbf{v},\mathbf{w}) \mapsto \mathbf{v}\dot{\times}\mathbf{w}$. The non-zero brackets of ordered pairs of elements of the considered admissible basis are

$$\mathbf{e}_x\dot{\times}\mathbf{e}_y = -\mathbf{e}_y\dot{\times}\mathbf{e}_x = \zeta\mathbf{e}_z\,,\ \mathbf{e}_y\dot{\times}\mathbf{e}_z = -\mathbf{e}_z\dot{\times}\mathbf{e}_y = \mathbf{e}_x\,,\ \mathbf{e}_z\dot{\times}\mathbf{e}_x = -\mathbf{e}_x\dot{\times}\mathbf{e}_z = \mathbf{e}_y\,.$$

For the Lie-Poisson structure of $\mathbf{F}$, the non-zero brackets of ordered pairs of coordinate functions are

$$\{x,y\} = -\{y,x\} = z\,,\ \{y,z\} = -\{z,y\} = \zeta x\,,\ \{z,x\} = -\{x,z\} = \zeta y\,,$$

and the expression of the Poisson bivector $\Lambda_{\mathbf{F}}$, in these coordinates, is

$$\Lambda_{\mathbf{F}}(x,y,z) = z\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y} + \zeta x\frac{\partial}{\partial y}\wedge\frac{\partial}{\partial z} + \zeta y\frac{\partial}{\partial z}\wedge\frac{\partial}{\partial x}\,.$$

Still in the coordinates functions considered, the expression of the pseudo-Riemannian or Riemannian metric on $\mathbf{F}$ determined by its scalar product is

$$\mathrm{d}s_{\mathbf{F}}{}^2(x,y,z) = \mathrm{d}x^2 + \mathrm{d}y^2 + \zeta\mathrm{d}z^2\,.$$

2.5. Coadjoint orbits of G as submanifolds of F

Since $\mathbf{F} \equiv \mathfrak{g}^*$, the coadjoint orbits of G can be considered as submanifolds of $\mathbf{F}$. So considered they are the connected submanifolds of $\mathbf{F}$ defined as

$$\{\mathbf{v}\in\mathbf{F}|\mathbf{v}\cdot\mathbf{v} = \text{Constant}\}\,,$$

with any possible Constant $\in\mathbb{R}$. In other words, with the coordinate functions x, y and z in an admissible basis $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ of $\mathbf{F}$, coadjoint orbits are connected submanifolds of $\mathbf{F}$ determined by an equation

$$x^2 + y^2 + \zeta z^2 = \text{Constant}\,,$$

for some Constant $\in\mathbb{R}$. The singleton $\{0\}$, whose unique element is the origin of $\mathbf{F}$, is a zero-dimensional coadjoint orbit. All other coadjoint orbits are two-dimensional.

Let $\mathcal{O}$ be any two-dimensional coadjoint orbit. On suitably chosen open subsets of $\mathcal{O}$, one can use as coordinates two of the three coordinate functions x, y and z associated with an admissible basis $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ of $\mathbf{F}$, the third coordinate being on the chosen subset a smooth function of the other two coordinates. Another possible choice of coordinates, which seems the most convenient, is made of the third coordinate z and the angular coordinate φ, defined by the equalities

$$x = \sqrt{x^2 + y^2}\cos\varphi\,, \quad y = \sqrt{x^2 + y^2}\sin\varphi\,.$$

The symplectic form $\omega_{\mathcal{O}}$ of the coadjoint orbit $\mathcal{O}$ admits the four equivalent expressions in terms of these coordinates:

$$\omega_{\mathcal{O}} = \frac{1}{z(x,y)}\mathrm{d}x \wedge \mathrm{d}y = \frac{\zeta}{x(y,z)}\mathrm{d}y \wedge \mathrm{d}z = \frac{\zeta}{y(z,x)}\mathrm{d}z \wedge \mathrm{d}x = \zeta \mathrm{d}\varphi \wedge \mathrm{d}z\,.$$

Each of the first three expressions of $\omega_{\mathcal{O}}$ is valid on open subsets of $\mathcal{O}$ on which the coordinate considered as a smooth function of the other two coordinates is non-zero: $z(x,y)$ for the first, $x(y,z)$ for the second and $y(z,x)$ for the third expression. The fourth expression is valid on the dense open subset of $\mathcal{O}$ on which the angular coordinate φ can be locally defined, *i.e.*, on the complementary subset of the set of points in $\mathcal{O}$ where both $x = 0$ and $y = 0$. When $\zeta = 1$ this occurs only at two points of each two-dimensional coadjoint orbit. When $\zeta = -1$, it occurs nowhere on some two-dimensional coadjoint orbits (the one-sheeted hyperboloids denoted below by H_R and the light cones with their apex removed denoted below by C^+ and C^-), and at a single point for other coadjoint orbits (the pseudo-spheres denoted below by P_R^+ and P_R^-). For this reason the coordinate system made of z and φ is the most convenient for the determination of Gibbs states. With these coordinates, the expression of the Liouville measure $\lambda_{\omega_{\mathcal{O}}}$ is

$$\lambda_{\omega_{\mathcal{O}}}(\mathrm{d}\mathbf{v}) = \mathrm{d}z\,\mathrm{d}\varphi\,, \quad \mathbf{v} \in \mathcal{O} \text{ with coordinates } (z,\varphi)\,.$$

When $\zeta = 1$, all two-dimensional coadjoint orbits are spheres centered on the origin 0 of $\mathbf{F}$. Their radius can be any real $R > 0$. We will denote by S_R the sphere of radius R centered on 0. It should be observed that the symplectic form on the coadjoint orbit ω_{S_R} *is not* the area form on this sphere, since it is proportional to R, not to R^2. The area form of S_R is $R\omega_{S_R}$.

When $\zeta = -1$, there are three kinds of two-dimensional coadjoint orbits, described below.

- The orbits, denoted by P_R^+ and P_R^-, whose respective equations are

  $$z = \sqrt{R^2 + x^2 + y^2} \text{ for } P_R^+ \text{ and } z = -\sqrt{R^2 + x^2 + y^2} \text{ for } P_R^-\,,$$

  with $R > 0$. They are called *pseudo-spheres* of radius R. Each one is a sheet of a two-sheeted two-dimensional hyperboloid with the z axis as revolution axis. They are said to be *space-like* submanifolds of $\mathbf{F}$, since all their tangent vectors are space-like vectors.

- The orbits, denoted by H_R, defined by the equation

$$x^2 + y^2 = z^2 + R^2 \,, \quad \text{wit } R > 0 \,.$$

  Each of these orbits is a single-sheeted hyperboloid with the z axis as revolution axis. The tangent space at any point to such an orbit is a two-dimensional Minkowski vector space.

- The two orbits, denoted by C^+ and C^-, defined respectively by

$$z^2 = x^2 + y^2 \text{ and } z > 0 \,, \qquad z^2 = x^2 + y^2 \text{ and } z < 0 \,.$$

  They are the cones in $\mathbf{F}$ (without their apex, the origin 0 of $\mathbf{F}$), made of light-like vectors directed, respectively, towards the future and towards the past.

3. Gibbs states on some two-dimensional symplectic manifolds

This section describes several examples of Gibbs states and the associated thermodynamic functions. Except those presented in the last Subsection (3.6), the results presented below rest on the properties of oriented three-dimensional Euclidean or pseudo-Euclidean vector spaces given in Section 2. Gibbs states on two-dimensional spheres, for the Hamiltonian action of the group of rotations $\mathrm{SO}(3)$, are presented in Subsection 3.1. Then in Subsection 3.2, the possible existence of Gibbs states on two-dimensional coadjoint orbits of the three-dimensional Lorentz group $\mathrm{SO}(2,1)$ is discusseded. It is proven that Gibbs states exist on the pseudopheres P_R^+ and P_R^-, but cannot exist on the other two-dimensional coadjoint orbits, the one-sheeted hyperboloids H_R and the light cones C^+ and C^-, the subset of generalized temperatures for these coadjoint orbits being empty. Using symplectomorphisms of the pseudospheres built with appropriate Möbius transformations, the Gibbs states and the associated thermodynamic functions for the Hamiltonian action of $\mathrm{SU}(1,1)$ on the Poincaré disk and of $\mathrm{SL}(2,\mathbb{R})$ on the Poincaré half-plane are determined, in Subsections 3.3 and 3.4, respectively. In Subsection 3.5, it is proven that no Gibbs state can exist on a two-dimensional symplectic vector space, for the Hamiltonian action of $\mathrm{SL}(2,\mathbb{R})$.

Finally, in 3.6, the Gibbs states and the associated thermodynamic functions for the Hamiltonian action, on an Euclidean affine and symplectic plane, of the group of its displacements, are determined.

3.1. Gibbs states on two-dimensional spheres

We assume here that $\zeta = 1$. The Lie group G is therefore isomorphic to $\mathrm{SO}(3)$ and its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(3)$. Let us consider the sphere S_R of radius $R > 0$ centered on the origin 0 of the vector space $\mathbf{F} \equiv \mathfrak{g}^*$ (identified with the dual $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$, as explained at the end of Subsection 2.2). This sphere is a coadjoint orbit, and the moment map of the Hamiltonian action of G on it is its canonical injection into $\mathbf{F} \equiv \mathfrak{g}^*$.

Proposition 1 *With the above indicated assumptions and notations, the open subset Ω of generalized temperatures, for the Hamiltonian action of $G \equiv \mathrm{SO}(3)$ on its coadjoint orbit S_R, is the whole Lie algebra $\mathfrak{g}$. For each $\boldsymbol{\beta} \in \mathbf{F} \equiv \mathfrak{g}$, we can choose an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$ such that $\mathbf{e}_z$ and $\boldsymbol{\beta}$ are parallel and directed in the same direction. Then $\boldsymbol{\beta} = \beta\mathbf{e}_z$, with $\beta \geq 0$. The partition function P and the probability density $\rho_{\boldsymbol{\beta}}$ of the Gibbs state indexed by $\boldsymbol{\beta}$ are expressed as*

$$P(\boldsymbol{\beta}) = \begin{cases} \dfrac{4\pi\sinh(R\beta)}{\beta} & \textit{if } \beta > 0, \\ 4\pi R & \textit{if } \beta = 0, \end{cases}$$

$$\rho_{\boldsymbol{\beta}}(\mathbf{r}) = \begin{cases} \dfrac{\beta\exp(-\beta z)}{4\pi\sinh(R\beta)} & \textit{if } \beta > 0, \\ \dfrac{1}{4\pi R} & \textit{if } \beta = 0, \end{cases} \qquad \textit{with } \mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \in S_R\,.$$

When $\beta > 0$, the thermodynamic functions mean value of J and entropy are

$$E_J(\boldsymbol{\beta}) = \frac{1 - R\beta\coth(R\beta)}{\beta^2}\,\boldsymbol{\beta}\,,$$
$$S(\boldsymbol{\beta}) = 1 + \log\left(\frac{4\pi\sinh(R\beta)}{\beta}\right) - R\beta\coth(R\beta)\,.$$

Proof: For each $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \in S_R$, we have

$$\bigl\langle J(\mathbf{r}), \boldsymbol{\beta}\bigr\rangle = \mathbf{r}\cdot\boldsymbol{\beta} = \beta z\,.$$

As explained in Subsection 2.5, on the dense open subset of S_R complementary to the union of the two poles $\{-R\mathbf{e}_z, R\mathbf{e}_z\}$, we can use the coordinate system (z, φ) and write

$$\int_{S_R}\exp\bigl(-\langle J(\mathbf{r}), \boldsymbol{\beta}\rangle\bigr)\lambda_{\omega_{S_R}}(\mathrm{d}\mathbf{r}) = \int_0^{2\pi}\left(\int_{-R}^{R}\exp(-\beta z)\mathrm{d}z\right)\mathrm{d}\varphi$$
$$= \begin{cases} 4\pi R & \text{if } \beta = 0, \\ \dfrac{4\pi\sinh(R\beta)}{\beta} & \text{if } \beta > 0. \end{cases}$$

Since S_R is compact, the above integral is always normally convergent. Therefore $\boldsymbol{\beta}$ is a generalized temperature. The stated results follow by easy calculations. ■

3.2. Gibbs states on pseudo-spheres and other $\mathrm{SO}(2,1)$ coadjoint orbits

We assume here that $\zeta = -1$. The Lie group G is therefore isomorphic to $\mathrm{SO}(2,1)$ and its Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{so}(2,1)$. For each coadjoint orbit $\mathcal{O}$ of G, we must determine whether the integral, which defines a function of the variable $\boldsymbol{\beta} \in \mathfrak{g}$,

$$\int_{\mathcal{O}} \exp\bigl(-\langle J(\mathbf{r}), \boldsymbol{\beta}\rangle\bigr)\lambda_{\omega_{\mathcal{O}}}\,\mathrm{d}\mathbf{r} \tag{$*$}$$

is normally convergent.

Proposition 2 *The considered coadjoint orbit $\mathcal{O}$ here is the pseudo-sphere P_R^+ defined in Subsection 2.5, for some $R > 0$. The set Ω of generalized temperatures is the subset of $\mathbf{F} \equiv \mathfrak{g}$ made of time-like vectors directed towards the past. For each $\boldsymbol{\beta} \in \Omega$, let $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an admissible basis of $\mathbf{F}$ such that $\boldsymbol{\beta} = \beta\mathbf{e}_z$, with $\beta < 0$. The partition function P and the probability density $\rho_{\boldsymbol{\beta}}$ of the Gibbs state indexed by $\boldsymbol{\beta}$, with respect to the Liouville measure $\lambda_{\omega_{P_{R+}}}$, are given by the formulae*

$$P(\boldsymbol{\beta}) = \frac{2\pi}{\|\boldsymbol{\beta}\|}\exp\bigl(-\|\boldsymbol{\beta}\|R\bigr)\,, \quad \boldsymbol{\beta} \in \mathbf{F}\,,\ \boldsymbol{\beta} \text{ time-like directed towards the past}\,,$$

$$\rho_{\boldsymbol{\beta}}(\mathbf{r}) = \frac{\|\boldsymbol{\beta}\|\exp\Bigl(-\|\boldsymbol{\beta}\|\bigl(z(\mathbf{r}) - R\bigr)\Bigr)}{2\pi}\,, \quad \mathbf{r} \in P_R^+\,,$$

where we have set $\|\boldsymbol{\beta}\| = \sqrt{-\boldsymbol{\beta}\cdot\boldsymbol{\beta}}$, since $\boldsymbol{\beta}\cdot\boldsymbol{\beta} < 0$.

The thermodynamic functions mean value of J and entropy are

$$E_J(\boldsymbol{\beta}) = -\frac{1 + R\|\boldsymbol{\beta}\|}{\|\boldsymbol{\beta}\|^2}\,\boldsymbol{\beta}\,,$$
$$S(\boldsymbol{\beta}) = 1 + \log\frac{2\pi}{\|\boldsymbol{\beta}\|}\,.$$

Proof: Let us first assume that $\boldsymbol{\beta} \in \mathbf{F} \equiv \mathfrak{g}$ is a non-zero time-like vector. There exists an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$ such that $\boldsymbol{\beta} = \beta\mathbf{e}_z$, with $\beta \in \mathbb{R}$, $\beta \neq 0$. For each $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \in \mathcal{O} \subset \mathbf{F} \equiv \mathbf{F}^*$, we can write

$$\bigl\langle J(\mathbf{r}), \boldsymbol{\beta}\bigr\rangle = \mathbf{r}\cdot\boldsymbol{\beta} = \zeta z\beta = -z\beta\,,$$

since $\zeta = -1$. We can choose (z, φ) as coordinates on the dense open subset of $\mathcal{O}$ complementary to the singleton $\{R\mathbf{e}_z\}$, so the above integral $(*)$ becomes

$$\int_0^{2\pi} \left(\int_R^{+\infty} \exp(\beta z) \mathrm{d}z \right) \mathrm{d}\varphi \,.$$

This integral is normally convergent if and only if $\beta < 0$, in other words if and only if the time-like vector $\boldsymbol{\beta}$ is directed towards the past.

Let us now assume that the vector $\boldsymbol{\beta}$ is space-like. We choose an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$ such that $\mathbf{e}_x$ and $\boldsymbol{\beta}$ are parallel. We therefore have $\boldsymbol{\beta} = \beta \mathbf{e}_x$, with $\beta \in \mathbb{R}$, $\beta \neq 0$. With (z, φ) as coordinate system, the above integral $(*)$ is expressed as

$$\int_0^{2\pi} \left(\int_R^{+\infty} \exp\left(-\beta \cos\varphi \sqrt{z^2 - R^2} \right) \mathrm{d}z \right) \mathrm{d}\varphi \,.$$

Using the fact that for $z > 0$ large enough, $\sqrt{z^2 - R^2} \equiv z$, we see that this integral is always divergent, as well when $\beta < 0$ as when $\beta > 0$, since $-\beta \cos\varphi > 0$ for many values of φ.

The subset Ω of $\mathbf{F} \equiv \mathfrak{g}$ of generalized temperatures contains all time-like vectors in $\mathbf{F}$ directed towards the past, no time-like vector directed towards the future and no space-like vector. Since it is open, it cannot contain the origin of $\mathbf{F}$, nor light-like vectors. Therefore Ω is exactly the subset of $\mathbf{F}$ made of time-like vectors directed towards the past. The expressions of the partition function P, the probability density $\rho_{\boldsymbol{\beta}}$, the mean value E_J of J and the entropy S given in the statement, follow by easy calculations. ■

Remark 3 Similarly, one can prove that when the considered coadjoint orbit $\mathcal{O}$ is the pseudo-sphere P_R^-, with $R > 0$, the open subset Ω of generalized temperatures is the subset of $\mathbf{F}$ made of time-like vectors directed towards the future. The probability density of Gibbs states and the corresponding thermodynamic functions are given by the same formulae as those indicated above, of course with the appropriate sign changes.

Proposition 4 *The coadjoint orbit $\mathcal{O}$ considered here is either a one-sheeted hyperboloid H_R, for some $R > 0$, or one of the light cones C^+ and C^-, defined in Subsection 2.5. The set Ω of generalized temperatures is empty. Therefore no Gibbs state, for the coadjoint action of the Lie group $G \equiv \mathrm{SO}(2,1)$ can exist on these coadjoint orbits.*

Proof: By calculations similar to those made in the part of the proof of Proposition 2 in which $\boldsymbol{\beta}$ is a space-like vector, it easily follows that for these coadjoint orbits, the set Ω of generalized temperatures is empty. ■

3.3. Gibbs states on the Poincaré disk

Assumptions and notations here are still those of 2, with $\zeta = -1$.

Proposition 5 *The choice of any admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$ determines, for each $R > 0$, a diffeomorphism ψ_R of the pseudo-sphere P_R^+ onto the Poincaré disk D_P, subset of the complex plane $\mathbb{C}$ whose elements w satisfy $|w| < 1$. Its expression is*

$$\psi_R(\mathbf{r}) = \frac{x+iy}{R+\sqrt{R^2+x^2+y^2}}\,, \qquad \mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + \sqrt{R^2+x^2+y^2}\,\mathbf{e}_z \in P_R^+\,.$$

The Poincaré disk D_P is endowed with a Riemannian metric ${\mathrm{d}s_{D_P}}^2$ and with a symplectic form ω_{D_P} for which ψ_R is both an isometry and a symplectomorphism. Their expressions are

$$\begin{aligned}
{\mathrm{d}s_{D_P}}^2(w) &= \frac{4R^2}{(1-|w|^2)^2}\,\mathrm{d}w\mathrm{d}\overline{w} = \frac{4R^2}{(1-|w|^2)^2}(\mathrm{d}w_{\mathrm{r}}^2 + \mathrm{d}w_{\mathrm{im}}^2)\,,\\
\omega_{D_P}(w) &= \frac{2iR}{(1-|w|^2)^2}\mathrm{d}w\wedge\mathrm{d}\overline{w} = \frac{4R}{(1-|w|^2)^2}\mathrm{d}w_{\mathrm{r}}\wedge\mathrm{d}w_{\mathrm{im}}\,,
\end{aligned}$$

w_{r} and w_{im} being the real and the imaginary parts of $w = w_{\mathrm{r}} + iw_{\mathrm{im}}$, respectively.

Proof: Let $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an admissible basis of $\mathbf{F}$, and let $R \in \mathbb{R}$ be a real satisfying $R > 0$. Let $\mathbf{E}$ be the vector subspace of $\mathbf{F}$ generated by $\mathbf{e}_x$ and $\mathbf{e}_y$. The *stereographic projection* of the pseudo-sphere P_R^+ on the subspace E, using $-R\mathbf{e}_z$ as pole of projection, is the map, defined on P_R^+, with values in $\mathbf{E}$, whose expression is

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + \sqrt{R^2+x^2+y^2}\,\mathbf{e}_z \mapsto \frac{R}{R+\sqrt{R^2+x^2+y^2}}(x\mathbf{e}_x + y\mathbf{e}_y)\,.$$

This map is smooth and injective, and its image is the open disk of radius R in $\mathbf{E}$ centered on the origin. When considered as taking its values in this disk, the stereographic pojection is a smooth diffeomorphism whose inverse, defined when $u^2+v^2<R^2$, is

$$u\mathbf{e}_x + v\mathbf{e}_\mathbf{y} \mapsto \frac{R}{R^2-(u^2+v^2)}\bigl(2R(u\mathbf{e}_x + v\mathbf{e}_\mathbf{y}) + (R^2+u^2+v^2)\mathbf{e}_z\bigr)\,.$$

Fore more details about stereographic projections, the reader is referred to [6] or to [9].

The map ψ_R, composed of the above described stereographic projection with the map

$$(u\mathbf{e}_x + v\mathbf{e}_y) \mapsto w = \frac{u+iv}{R}\,,$$

is therefore a diffeomorphism of the pseudo-sphere P_R^+ onto the Poincaré disk D_P. The pseudo-sphere P_R^+ is endowed both with the Riemannian metric induced by that of $\mathbf{F}$, and with its symplectic form of coadjoint orbit of the Lie group $G \equiv \mathrm{SO}(2,1)$ ($\mathbf{F}$ being identified with the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g} \equiv \mathfrak{so}(2,1)$). The Poincaré disk D_P is therefore endowed with a Riemannian metric $\mathrm{d}{s_{D_P}}^2$ and with a symplectic form ω_{D_P} for which the map ψ_R is both an isometry and a symplectomorphism. Their expressions, given in the statement, easily follow from those of the Riemannian metric and the symplectic form of P_R^+, indicated in Subsection 2.5. The real $R > O$ plays the part, in the expression of the Riemannian metric $\mathrm{d}{s_{D_P}}^2$, of a unit of length chosen on D_P. ■

Proposition 6 *The Lie group $G \equiv \mathrm{SO}(2,1)$ acts, by a Hamiltonian action, on the Poincaré disk D_P endowed with its symplectic form ω_{D_P}. The map J_{D_P}, defined on D_P, taking its values in $F \equiv \mathfrak{g}^*$,*

$$J_{D_P}(w) = \frac{R}{1-|w|^2}\bigl(2(w_\mathrm{r}\mathbf{e}_x + w_\mathrm{im}\mathbf{e}_y) + (1+|w|^2)\mathbf{e}_z\bigr)\,,$$

is a moment map of this action. In the above expression, $w = w_\mathrm{r} + iw_\mathrm{im} \in D_p \subset \mathbb{C}$, $|w|^2 = w_\mathrm{r}^2 + w_\mathrm{im}^2 < 1$.

Proof: The vector space $\mathbf{F}$ being identified with the dual vector space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g} \equiv \mathfrak{so}(2,1)$ as explained in Subsection 2.2, its submanifold P_R^+ is identified with a coadjoint orbit of $G \equiv \mathrm{SO}(2,1)$. The coadjoint action of G, restricted to $\mathbf{F}$ endowed with its symplectic form of coadjoint orbit, is therefore Hamiltonian and admits the canonical injection of P_R^+ in $\mathfrak{g}^* \equiv \mathbf{F}$ as a Hamiltonian. The stated result therefore follows from Proposition 5, according to which ψ_R is a symplectomorphism of P_R^+, endowed with its symplectic form of coadjoint orbit, onto D_P endowed with the symplectic form ω_{D_P}. The stated expression of J_{D_P} easily follows from that of the inverse of the stereographic projection indicated in the proof of Proposition 5. ■

Remark 7 On the open dense subset of D_P complementary to the singleton $\{0\}$, the polar coordinate φ such that

$$w_\mathrm{r} = |w|\cos\varphi\,,\quad w_\mathrm{im} = |w|\sin\varphi\,,\quad \text{with } |w| = \sqrt{w_\mathrm{r}^2 + w_\mathrm{im}^2}\,,$$

can be locally defined. Then we can write

$$\mathrm{d}w_{\mathrm{r}} \wedge \mathrm{d}w_{\mathrm{im}} = |w|\mathrm{d}|w| \wedge \mathrm{d}\varphi\,,$$

so $\omega_{D_P}(w)$ can be expressed as

$$\omega_{D_P}(w) = \frac{4R|w|}{(1-|w|^2)^2}\mathrm{d}|w| \wedge \mathrm{d}\varphi = \mathrm{d}\left(\frac{2R}{1-|w|^2}\right) \wedge \mathrm{d}\varphi\,.$$

Therefore on this dense open subset of (D_P, ω_{D_P}), the Liouville measure $\lambda_{\omega_{D_P}}$, expressed in terms of the local polar coordinates $(|w|, \varphi)$, is

$$\lambda_{\omega_{D_P}}(\mathrm{d}w) = \frac{4R|w|}{(1-|w|^2)^2}\mathrm{d}|w|\mathrm{d}\varphi\,.$$

Proposition 8 *The set Ω of generalized temperatures, for the Hamiltonian action of $G \equiv \mathrm{SO}(2,1)$ on the Poincaré disk D_P endowed with its symplectic form ω_{D_P}, is the subset of the Lie algebra $\mathfrak{g} \equiv \mathfrak{so}(2,1) \equiv \mathbf{F}$ made of timelike vectors directed towards the past. Let $\boldsymbol{\beta} \in \Omega$, and $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ be an admissible basis of $\mathbf{F}$ such that $\boldsymbol{\beta} = \beta\mathbf{e}_z$, with $\beta \in \mathbb{R}$, $\beta < 0$. We identify D_P wih P_R^+ by means of the diffeomorphism ψ_R built with this admissible basis, as explained in Proposition 5. On the open dense subset of D_P complementary to the singleton $\{0\}$, the probability density of the Gibbs state indexed by $\boldsymbol{\beta}$, with respect to the Liouville measure $\lambda_{\omega_{D_P}}$, is*

$$\rho_{\boldsymbol{\beta}}(w) = \frac{|\beta|}{2\pi}\exp\left(-\frac{2R|\beta|}{1-|w|^2}\right)\,, \quad \textit{with } w \in D_P\,.$$

The associated thermodynamic functions (mean value of the moment map E_J and entropy S) are the functions of the generalized temperature $\boldsymbol{\beta}$ whose expressions are given in Proposition 2 in Subsection 3.2.

Proof: The sets of generalized temperatures for the Hamiltonian actions of G on the Poincaré disk D_P and on the pseudo-sphere P_R^+ are the same, since the map ψ_R built with any admissible basis of $\mathbf{F}$ is a symplectomorphism equivariant with respect to the actions of G on these two symplectic manifolds. Therefore Ω is, as seen in Proposition 2 in Subsection 3.2, the subset of the Lie algebra $\mathfrak{g}$ made of timelike vectors directed towards the past. According to this Proposition, on the pseudo-sphere P_R^+, the probability density of the Gibbs state indexed by the generalized temperature $\boldsymbol{\beta} = \beta\mathbf{e}_z$, ($\beta < 0$), with respect to the Liouville measure $\lambda_{\omega_{P_R^+}}$, is

$$\rho_{\boldsymbol{\beta}}(\mathbf{r}) = \frac{|\beta|\exp\Big(-|\beta|\big(z(\mathbf{r}) - R\big)\Big)}{2\pi}\,, \quad \mathbf{r} \in P_R^+\,.$$

On the Poincaré disk D_P, the probability density of the Gibbs state indexed by $\boldsymbol{\beta}$, with respect to the Liouville measure $\lambda_{\omega_{D_P}}$, is deduced from $\rho_{\boldsymbol{\beta}}(\mathbf{r})$ by replacing $z(\mathbf{r})$ by its expression in terms of w, deduced from the expression of the inverse of the stereographic projection given in the proof of Proposition 5 in this Subsection :

$$z(\mathbf{r}) = \frac{R(1+|w|^2)}{1-|w|^2}\,, \quad \text{therefore} \quad z(\mathbf{r}) - R = \frac{2R}{1-|w|^2}\,.$$

The expression of $\rho_{\boldsymbol{\beta}}(w)$ indicated in the statement easily follows. Of course the thermodynamic functions are the same as those for the corresponding Gibbs state on P_R^+. ■

Remark 9 We have seen (Remark 7 in this Subsection) that on the open dense subset of D_P on which $w \neq 0$, the Liouville measure $\lambda_{\omega_{D_P}}$, expressed in terms of the local polar coordinates $(|w|, \varphi)$, is

$$\lambda_{\omega_{D_P}}(\mathrm{d}w) = \frac{4R|w|}{(1-|w|^2)^2}\mathrm{d}|w|\mathrm{d}\varphi\,.$$

Using this expression, we see that the probability density of the Gibbs state indexed by $\boldsymbol{\beta} = \beta\mathbf{e}_z$, with $\beta < 0$, with respect to the measure $\mathrm{d}|w|\mathrm{d}\varphi$, is

$$\frac{4R|w|}{(1-|w|^2)^2}\rho_{\boldsymbol{\beta}}(w) = \frac{2R|\beta||w|}{\pi(1-|w|^2)^2}\exp\left(-\frac{2R|\beta|}{1-|w|^2}\right)\,, \quad \textit{with } w \in D_P\,.$$

Remark 10 Use of the Möbius transformation Instead of the Lie group $G \equiv \mathrm{SO}(2,1)$, the Lie group $\mathrm{SU}(1,1)$ is very often used as group of symmetries of the Poincaré disk D_P. It is the group of complex 2×2 matrices wich can be written as

$$A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix}\,, \quad \text{with } a \text{ and } b \in \mathbb{C}\,,\ |a|^2 - |b|^2 = a\overline{a} - b\overline{b} = 1\,. \qquad (*)$$

This group acts on the Poincaré disk D_P by *Möbius transformations*, so called in honour of the German mathematician August Ferdinand Möbius (1790–1868). We recall that the Möbius transformation determined by a complex 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a, b, c and $d \in \mathbb{C}$ satisfying $ad - bc \neq 0$, is the map $U_A : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, with $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$U_A(w) = \begin{cases} \dfrac{aw+b}{cw+d} & \text{if } w \in \mathbb{C} \text{ and } cw+d \neq 0,\\ \infty & \text{if } w \in \mathbb{C} \text{ and } cw+d = 0,\\ \dfrac{a}{c} & \text{if } w = \infty \text{ and } c \neq 0,\\ \infty & \text{if } w = \infty \text{ and } c = 0.\end{cases}$$

The Möbius transformations U_A and $U_{A'}$ determined by the two matrices A and A' are equal if and only if $A' = \lambda A$ for some $\lambda \in \mathbb{C}$, $\lambda \neq 0$. When A and $A' \in \mathrm{SU}(1,1)$, $U_{A'} = U_A$ if and only if $A' = \pm A$. The Möbius transformation U_A, determined by $A \in \mathrm{SU}(1,1)$, restricted to the Poincaré disk D_P, is a diffeomorphism of D_P onto itself, and the map

$$\Xi : \mathrm{SU}(1,1) \times D_P \to D_P\,, \quad \Xi(A,w) = U_A(w)$$

so defined is a holomorphic Hamiltonian action of $\mathrm{SU}(1,1)$ on D_P, endowed with its symplectic form ω_{D_P}. There exists a surjective Lie groups homomorphism Φ of $\mathrm{SU}(1,1)$ onto $\mathrm{SO}(2,1)$ whose kernel is the discrete group $\{1,-1\}$ (where 1 stands for the unit 2×2 matrix and -1 for the opposite matrix). For each complex 2×2 matrix $A \in \mathrm{SU}(1,1)$, expressed as indicated by the formulae $(*)$ above, $\Phi(A)$ is the real 3×3 matrix (see for example [6])

$$\Phi(A) = \begin{pmatrix} \dfrac{a^2 + \overline{a}^2 + (b^2 + \overline{b}^2)}{2} & -\dfrac{a^2 - \overline{a}^2 - (b^2 - \overline{b}^2)}{2i} & -(ab + \overline{ab}) \\ \dfrac{a^2 - \overline{a}^2 + (b^2 - \overline{b}^2)}{2i} & \dfrac{a^2 + \overline{a}^2 - (b^2 + \overline{b}^2)}{2} & -\dfrac{ab - \overline{ab}}{i} \\ -(a\overline{b} + \overline{a}b) & \dfrac{-(\overline{a}b - a\overline{b})}{i} & (a\overline{a} + b\overline{b}) \end{pmatrix}. \quad (**)$$

The map $T_e\Phi : T_e\mathrm{SU}(1,1) \to T_e\mathrm{SO}(2,1)$, where e stands for the unit elements of both $\mathrm{SU}(1,1)$ and $\mathrm{SO}(2,1)$, is a Lie algebras isomorphism, with the usual convention of identifying the Lie algebra of a Lie group with the tangent space to its unit element. The Lie algebras of the Lie groups $\mathrm{SU}(1,1)$ and $\mathrm{SO}(2,1)$ are therefore isomorphic, and can both be identified with the vector space $\mathbf{F}$, as well as their dual vector spaces. The action Ξ of $\mathrm{SU}(1,1)$ on the Poincaré disk D_P by Möbius transformations can therefore be identified with the Hamiltonian action of $G \equiv \mathrm{SO}(2,1)$ discussed above, and admits J_{D_P} as moment map.

3.4. Gibbs states on the Poincaré half-plane

The Poincaré half-plane is a well known model of non-Euclidean geometry, equivalent to the Poincaré disk D_P. We recall in the present subsection how all the results obtained above for the Poincaré disk D_P can be applied to the Poincaré half-plane.

Proposition 11 *The map $\chi : D_P \to \mathbb{C}$, defined on the Poincaré disk D_P and taking its values in $\mathbb{C}$,*

$$\chi(w) = \frac{i(-w+1)}{w+1}\,, \quad w \in D_P = \{w \in \mathbb{C} \mid |w| < 1\}\,,$$

is smooth and injective. Its image is the Poincaré half-plane

$$\Pi_P = \{\xi = \xi_{\rm r} + i\xi_{\rm im} \in \mathbb{C} \,|\, \xi_{\rm im} > 0\}\,,$$

where $\xi_{\rm r}$ *and* $\xi_{\rm im}$ *are respectively the real and the imaginary parts of the complex number* ξ*. Considered as taking its values in* Π_P*, this map is a a diffeomorphism of the Poincaré disk* D_P *onto the Poincaré half-plane* Π_P*. The Poincaré half-plane* Π_P *can be endowed with a Riemannian metric* ${\rm d}s_{\Pi_P}{}^2$ *and with a symplectic form* ω_{Π_P} *for which the map* $w \mapsto \xi$ *is both an isometry and a symplectomorphism. Their expressions are*

$$\begin{aligned} {\rm d}s_{\Pi_P}{}^2(\xi) &= \frac{R^2}{\xi_{\rm im}^2}({\rm d}\xi_{\rm r}^2 + {\rm d}\xi_{\rm im}^2)\,,\\ \omega_{\Pi_P}(\xi) &= \frac{R}{\xi_{\rm im}^2}{\rm d}\xi_{\rm r}\wedge {\rm d}\xi_{\rm im}\,. \end{aligned}$$

Proof: With the notations of Remark 10 in Subsection 3.3, the map $w \mapsto \xi$ is the Möbius transformation U_M determined by the matrix $M = \begin{pmatrix} -i & i\\ 1 & 1\end{pmatrix}$, restricted to the Poincaré disk D_P. One can easily check that it is smooth and injective, that its image is the Poincaré half-plane Π_P and that when considered as taking its values in Π_P, it is a smooth diffeomorphism of D_P onto Π_P. Using this diffeomorphism, the Riemannian metric and the symplectic form of D_P can be transported on Π_P. The expressions of the Riemannian metric ${\rm d}s_{\Pi_P}{}^2$ and of the symplectic form ω_{Π_P} so obtained on Π_P, indicated in the statement, easily follow from those of the Riemannian metric and of the symplectic form on D_P, indicated in Proposition 5 of Subsection 3.3. ■

Proposition 12 *For each* 2×2 *matrix with complex coefficients*

$$A = \begin{pmatrix} a & b\\ \overline{b} & \overline{a}\end{pmatrix} \in {\rm SU}(1,1)\,, \quad \textit{with } a \textit{ and } b \in \mathbb{C}\,,\ |a|^2 - |b|^2 = a\overline{a} - b\overline{b} = 1\,,$$

let $\Sigma(A)$ *be the* 2×2*-matrix with real coefficients*

$$\Sigma(A) = \begin{pmatrix} a_{\rm r} - b_{\rm r} & a_{\rm im} + b_{\rm im}\\ -a_{\rm im} + b_{\rm im} & a_{\rm r} + b_{\rm r}\end{pmatrix}\,,$$

where $a_{\rm r}$ *and* $a_{\rm im}$ *are, respectively, the real and the imaginary parts of the complex number* $a = a_{\rm r} + ia_{\rm im}$*, and where* $b_{\rm r}$ *and* $b_{\rm im}$ *are, respectively, the real and the imaginary parts of the complex number* $b = b_{\rm r} + ib_{\rm im}$*. The map* Σ *is a smooth Lie groups isomorphism of the Lie group* ${\rm SU}(1,1)$ *onto the Lie group* ${\rm SL}(2,\mathbb{R})$*.*

For each 2×2*-matrix* $N=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}(2,\mathbb{R})$*, that means such that the real numbers* α*,* β*,* γ *and* δ *satify* $\alpha\delta-\gamma\beta=1$*, and each* $\xi\in\mathbb{C}$*, we set*

$$\Psi(N,\xi)=U_N(\xi)=\frac{\alpha\xi+\beta}{\gamma\xi+\delta}\,,$$

where U_N *is the Möbius transformation associated with* N*, as explained in Remark 10 in Subsection 3.3. The map* $\Psi:\mathrm{SL}(2,\mathbb{R})\times\Pi_P\to\Pi_P$ *so defined is a Hamiltonian action of the Lie goup* $\mathrm{SL}(2,\mathbb{R})$ *on the Poincaré half-plane* Π_P*, endowed with its symplectic form* ω_{Π_P}*. This action is equivalent to the Hamiltonian action* Ξ *of* $\mathrm{SU}(1,1)$ *on the Poincaré disk* D_P*, which means that for each* $A\in\mathrm{SU}(1,1)$ *and* $\xi\in\Pi_P$*),*

$$\Psi\bigl(\Sigma(A),\xi\bigr)=\Xi\bigl(A,\chi^{-1}(\xi)\bigr)\,.$$

The dual vector space of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ *can be identified with the vector space* $\mathbf{F}$*. With this identification, the expression of the moment map of the Hamiltonian action of* $\mathrm{SL}(2,\mathbb{R})$ *on* Π_P *is*

$$J_{\Pi_P}(\xi)=\frac{R}{2\xi_{\mathrm{im}}}\bigl((1-|\xi|^2)\mathbf{e}_x+2\xi_{\mathrm{r}}\mathbf{e}_y+(1+|\xi|^2)\mathbf{e}_z\bigr)\,.$$

The set Ω *of generalized temperatures, for the Hamiltonian action* Ψ*, is the set of time-like elements in* $\mathbf{F}\equiv\mathfrak{sl}(2,\mathbb{R})$ *directed towards the past. Let* $\boldsymbol{\beta}$ *be one of its elements. We choose an admissible basis* $(\mathbf{e}_x,\mathbf{e}_y,\mathbf{e}_z)$ *of* $\mathbf{F}$ *such that* $\boldsymbol{\beta}=\beta\mathbf{e}_z$*, with* $\beta<0$*. The probability density* $\rho_{\boldsymbol{\beta}}$ *of the Gibbs state on* Π_P *indexed by* $\boldsymbol{\beta}$*, with respect to the Liouville measure* $\lambda_{\omega_{\Pi_P}}$*, is*

$$\rho_{\boldsymbol{\beta}}(\xi)=\frac{|\beta|}{2\pi}\exp\left(-\frac{R|\beta|\bigl((1+\xi_{\mathrm{im}})^2+\xi_{\mathrm{r}}^2\bigr)}{2\xi_{\mathrm{im}}}\right)\,,\quad \xi=\xi_{\mathrm{r}}+i\xi_{\mathrm{im}}\in\Pi_P\,.$$

The associated thermodynamic functions (mean value of the moment map E_J *and entropy* S*) are the functions of the generalized temperature* $\boldsymbol{\beta}$ *whose expressions are given in Proposition 2, Subsection 3.2.*

Proof: For any matrix $A=\begin{pmatrix}a & b\\ \overline{b} & \overline{a}\end{pmatrix}\in\mathrm{SU}(1,1)$, $\det\bigl(\Sigma(A)\bigr)=|a|^2-|b|^2=1$, which proves that $\Sigma(A)\in\mathrm{SL}(2,\mathbb{R})$. By direct calculations, one can check that Σ is indeed a smooth Lie groups isomorphism of $\mathrm{SU}(1,1)$ onto $\mathrm{SL}(2,\mathbb{R})$.

The matrix $A\in\mathrm{SU}(1,1)$ acts on the Poincaré disk D_P by the Möbius transformation U_A defined in Remark 10 of Subsection 3.3. Since, as seen in the proof of Proposition 11 in the present Subection, the Möbius transformation U_M determined by the matrix $M=\begin{pmatrix}-i & i\\ 1 & 1\end{pmatrix}$, restricted to D_P, is a diffeomorphim of

D_P onto the Poincaré half-plane Π_P, the corresponding action of A on Π_P is the Möbius transformation determined by the matrix

$$\begin{aligned} MAM^{-1} &= \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \begin{pmatrix} i/2 & 1/2 \\ -i/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} a+\overline{a}-b-\overline{b} & -i(a-\overline{a}+b-\overline{b}) \\ i(a-\overline{a}-b+\overline{b}) & a+\overline{a}+b+\overline{b} \end{pmatrix} \\ &= 2\begin{pmatrix} a_{\mathrm{r}}-b_{\mathrm{r}} & a_{\mathrm{im}}+b_{\mathrm{im}} \\ -a_{\mathrm{im}}+b_{\mathrm{im}} & a_{\mathrm{r}}+b_{\mathrm{r}} \end{pmatrix}. \end{aligned}$$

The Möbius transformations determined by MAM^{-1} and by $(1/2)MAM^{-1}$ being equal, we are led to consider the map Σ, defined on $\mathrm{SU}(1,1)$, taking its values in the set of real 2×2 matrices, which associates to each matrix $A\in\mathrm{SU}(1,1)$ the matrix

$$\Sigma(A) = \begin{pmatrix} a_{\mathrm{r}}-b_{\mathrm{r}} & a_{\mathrm{im}}+b_{\mathrm{im}} \\ -a_{\mathrm{im}}+b_{\mathrm{im}} & a_{\mathrm{r}}+b_{\mathrm{r}} \end{pmatrix}, \quad \text{with } A = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathrm{SU}(1,1).$$

The map Ψ defined in the statement is therefore a Hamiltonian action of $\mathrm{SL}(2,\mathbb{R})$ on Π_P, equivalent to the Hamiltonian action Ξ of $\mathrm{SU}(1,1)$ on D_P. As for the action of $\mathrm{SU}(1,1)$ on the Poincaré disk D_P, the dual vector space of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ can be identified with the vector space $\mathbf{F}$. With this identification, the expression of the moment map J_{Π_P} of the Hamiltonian action Ψ indicated in the statement follows from that of the moment map J_{D_P} of the Hamiltonian action of $\mathrm{SO}(2,1)$ on D_P (Proposition 6 in Subsection 3.3). Similarly, the other results in the statement follow from the corresponding results for the Hamiltonian action of $\mathrm{SU}(1,1)$ on D_P. ■

Remark 13 *The Liouville measure on the Poincaré half-plane Π_P, endowed with its symplectic form ω_{Π_P}, is easily deduced from that of the Liouville measure on the Poincaré disk D_P, indicated in Remark 7, Subsection 3.3. Its expression is*

$$\lambda_{\omega_{\Pi_P}}(\mathrm{d}\xi) = \frac{R}{\xi_{\mathrm{im}}^2}\mathrm{d}\xi_{\mathrm{r}}\mathrm{d}\xi_{\mathrm{im}}.$$

Therefore the expression of the probability density of the Gibbs state on Π_P indexed by the generalized temperature $\boldsymbol{\beta}=\beta\mathbf{e}_z$, with $\beta<0$, with respect to the measure $\mathrm{d}\xi_{\mathrm{r}}\mathrm{d}\xi_{\mathrm{im}}$*, is*

$$\frac{R}{\xi_{\mathrm{im}}^2}\rho_{\boldsymbol{\beta}}(\xi) = \frac{R|\beta|}{2\pi\xi_{\mathrm{im}}^2}\exp\left(-\frac{R|\beta|\big((1+\xi_{\mathrm{im}})^2+\xi_{\mathrm{r}}^2\big)}{2\xi_{\mathrm{im}}}\right), \quad \xi=\xi_{\mathrm{r}}+i\xi_{\mathrm{im}}\in\Pi_P.$$

3.5. No Gibbs state can exist on a two-dimensional symplectic vector space

We consider the plane $\mathbb{R}^2$ (coordinates u, v), endowed with the symplectic form $\omega = \mathrm{d}u \wedge \mathrm{d}v$. The symplectic group $\mathrm{Sp}(\mathbb{R}^2, \omega)$ is the group $\mathrm{SL}(2, \mathbb{R})$ of real 2×2 matrices with determinant 1. As seen in Subsection 2.2, its Lie algebra, as well as its dual vector space, can be identified with the vector space $\mathbf{F}$, once an admissible basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$ is chosen. The infinitesimal generators of the action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{R}^2$ are the three Hamiltonian vector fields

$$\begin{aligned}
X_{\mathbb{R}^2}(u,v) &= \frac{1}{2}\left(v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right), \text{ whose Hamiltonian is } H_{X_{\mathbb{R}^2}}(u,v) = \frac{u^2 - v^2}{4}\,,\\
Y_{\mathbb{R}^2}(u,v) &= \frac{1}{2}\left(u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}\right), \text{ whose Hamiltonian is } H_{Y_{\mathbb{R}^2}} = -\frac{uv}{2}\,,\\
Z_{\mathbb{R}^2}(u,v) &= \frac{1}{2}\left(v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}\right), \text{ whose Hamiltonian is } H_{Z_{\mathbb{R}^2}} = -\frac{u^2 + v^2}{4}\,.
\end{aligned}$$

The infinitesimal generators $X_{\mathbb{R}^2}$, $Y_{\mathbb{R}^2}$ and $Z_{\mathbb{R}^2}$ are the images, by the action on $\mathbb{R}^2$ of the Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \equiv \mathbf{F}$, of $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, respectively. We therefore obtain a moment map $J_{\mathbb{R}^2} : \mathbb{R}^2 \to \mathbf{F} \equiv \mathfrak{sl}(2, \mathbb{R})^*$ of this action by writing $\langle J_{\mathbb{R}^2}(u,v), \mathbf{e}_x \rangle = H_{X_{\mathbb{R}^2}}(u,v)$, $\langle J_{\mathbb{R}^2}(u,v), \mathbf{e}_y \rangle = H_{Y_{\mathbb{R}^2}}(u,v)$, $\langle J_{\mathbb{R}^2}(u,v), \mathbf{e}_z \rangle = H_{X_{\mathbb{R}^2}}(u,v)$. So we have

$$J_{\mathbb{R}^2}(u,v) = H_{X_{\mathbb{R}^2}}(u,v)\boldsymbol{\varepsilon}_x + H_{Y_{\mathbb{R}^2}}(u,v)\boldsymbol{\varepsilon}_y + H_{Z_{\mathbb{R}^2}}(u,v)\boldsymbol{\varepsilon}_z\,,$$

where $(\boldsymbol{\varepsilon}_x, \boldsymbol{\varepsilon}_y, \boldsymbol{\varepsilon}_z)$ is the basis of $\mathbf{F}^*$ dual of the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ of $\mathbf{F}$. With the identification of $\mathbf{F}$ with its dual $\mathbf{F}^*$ by means of the scalar product on $\mathbf{F}$ of signature $(+,+,-)$, we have $\boldsymbol{\varepsilon}_x = \mathbf{e}_x$, $\boldsymbol{\varepsilon}_y = \mathbf{e}_y$, $\boldsymbol{\varepsilon}_z = -\mathbf{e}_z$. Therefore

$$\begin{aligned}
J_{\mathbb{R}^2}(u,v) &= H_{X_{\mathbb{R}^2}}(u,v)\mathbf{e}_x + H_{Y_{\mathbb{R}^2}}(u,v)\mathbf{e}_y - H_{Z_{\mathbb{R}^2}}(u,v)\mathbf{e}_z\\
&= \frac{u^2 - v^2}{4}\mathbf{e}_x - \frac{uv}{2}\mathbf{e}_y + \frac{u^2 + v^2}{4}\mathbf{e}_z\,.
\end{aligned}$$

By observing that

$$\left(\frac{u^2 - v^2}{4}\right)^2 + \left(\frac{uv}{2}\right)^2 - \left(\frac{u^2 + v^2}{4}\right)^2 = 0 \quad \text{and} \quad \frac{u^2 + v^2}{4} \geq 0\,,$$

we see that the $J_{\mathbb{R}^2}(\mathbb{R}^2)$ is the union of two coadjoint orbits of $\mathrm{SL}(2, \mathbb{R})$: a zero-dimensional orbit, the singleton $\{0\}$ (where 0 stands for the origin of $\mathbf{F}$), and a two-dimensional orbit, the cone C^+ of light-like elements in $\mathbf{F}$ directed towards the future. We have seen above (Proposition 2, Subsection 3.2) that no Gibbs state can exist on C^+. Therefore no Gibbs state can exist on a two-dimensional symplectic vector space, for the natural action of the linear symplectic group.

3.6. Gibbs states on an affine Euclidean and symplectic plane

As in Subsection 3.5, we consider the plane $\mathbb{R}^2$ (coordinates u, v) endowed with the symplectic form $\omega = \mathrm{d}u \wedge \mathrm{d}v$. Moreover we endow it with its usual Euclidean metric, and consider the action of its group of displacements (rotations and translations), denoted by $\mathrm{E}(2,\mathbb{R})$. In matrix notations, an element of $\mathbb{R}^2$ of coordinates (u, v) is represented by the column vector $\begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$ and an element $g_{(\varphi,x,y)}$ of $\mathrm{E}(2,\mathbb{R})$ by a matrix $\begin{pmatrix} \cos\varphi & -\sin\varphi & x \\ \sin\varphi & \cos\varphi & y \\ 0 & 0 & 1 \end{pmatrix}$ depending on the three real parameters φ, x and y. The action of $\mathrm{E}(2,\mathbb{R})$ on $\mathbb{R}^2$ is expressed as the product of matrices

$$\begin{pmatrix} \cos\varphi & -\sin\varphi & x \\ \sin\varphi & \cos\varphi & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \begin{pmatrix} u\cos\varphi - v\sin\varphi + x \\ u\sin\varphi + v\cos\varphi + y \\ 1 \end{pmatrix} .$$

We denote by $(\mathbf{e}_r, \mathbf{e}_x, \mathbf{e}_y)$ the basis of the Lie algebra $\mathfrak{e}(2,\mathbb{R})$ whose elements, identified with the corresponding matrices, are

$$\mathbf{e}_r = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \mathbf{e}_x = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \mathbf{e}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} .$$

The corresponding fundamental vector fields on $\mathbb{R}^2$ are the Hamilonian vector fields, generators of the action of $\mathrm{E}(2,\mathbb{R})$,

$$(\mathbf{e}_r)_{\mathbb{R}^2}(u,v) = -v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v} , \quad \text{whose Hamiltonian is } H_{(\mathbf{e}_r)_{\mathbb{R}^2}}(u,v) = \frac{u^2+v^2}{2} ,$$
$$(\mathbf{e}_x)_{\mathbb{R}^2}(u,v) = \frac{\partial}{\partial u} , \quad \text{whose Hamiltonian is } H_{(\mathbf{e}_x)_{\mathbb{R}^2}}(u,v) = -v ,$$
$$(\mathbf{e}_y)_{\mathbb{R}^2}(u,v) = \frac{\partial}{\partial v} , \quad \text{whose Hamiltonian is } H_{(\mathbf{e}_y)_{\mathbb{R}^2}}(u,v) = u .$$

Proceeding as in Subsection 3.5, we obtain the expression of the moment map

$$J_{\mathbb{R}^2}(u,v) = \frac{u^2+v^2}{2}\boldsymbol{\varepsilon}_r - y\boldsymbol{\varepsilon}_x + x\boldsymbol{\varepsilon}_y ,$$

where $(\boldsymbol{\varepsilon}_r, \boldsymbol{\varepsilon}_x, \boldsymbol{\varepsilon}_y)$ is the basis of $\mathfrak{e}(2,\mathbb{R})^*$ dual of the basis $(\mathbf{e}_r, \mathbf{e}_x, \mathbf{e}_y)$ of $\mathfrak{e}(2,\mathbb{R})$. An element $\boldsymbol{\beta} = \beta_r\mathbf{e}_r + \beta_x\mathbf{e}_x + \beta_y\mathbf{e}_y$ in $\mathfrak{e}(2,\mathbb{R})$ is a generalized temperature if, considered as a function of β_r, β_x and β_y, the integral

$$\int_{\mathbb{R}^2} \exp\bigl(-\langle J(u,v), \boldsymbol{\beta}\rangle\bigr)\lambda_\omega = \int_{\mathbb{R}^2} \exp\left(-\frac{u^2+v^2}{2}\beta_r + v\beta_x - u\beta_y\right) \mathrm{d}u\mathrm{d}v \quad (*)$$

is normally convergent. Clearly, a necessary condition for the normal convergence of this integral is

$$\beta_r > 0\,. \tag{$**$}$$

When this condition is satisfied, we can write

$$-\frac{u^2+v^2}{2}\beta_r + v\beta_x - u\beta_y = \frac{\beta_x^2+\beta_y^2}{2\beta_r} - \frac{\beta_r}{2}\left[\left(u+\frac{\beta_y}{\beta_r}\right)^2 + \left(v-\frac{\beta_x}{\beta_r}\right)^2\right]\,.$$

By using on the plane $\mathbb{R}^2$, instead of (u,v), the polar coordinates (ρ,ψ), determined by

$$u' = u + \frac{\beta_y}{\beta_r} = \rho\cos\psi\,,\quad v' = v - \frac{\beta_x}{\beta_r} = \rho\sin\psi\,,$$

we see that when $(**)$ is satisfied, the integral $(*)$ above is normally convergent. Condition $(**)$ is therefore both necessary and sufficient for the normal convergence of $(*)$. The set Ω of generalized temperatures, for the action of $\mathrm{E}(2,\mathbb{R})$ on $(\mathbb{R}^2,\omega)$, is made of elements $\boldsymbol{\beta} = \beta_r\mathbf{e}_r + \beta_x\mathbf{e}_x + \beta_y\mathbf{e}_y \in \mathfrak{e}(2,\mathbb{R})$ which satisfy Condition $(**)$ above. The expression of the partition function P is then

$$\begin{aligned}P(\boldsymbol{\beta}) &= \exp\left(\frac{\beta_x^2+\beta_y^2}{2\beta_r}\right)\int_0^{2\pi}\left(\int_0^{+\infty}\exp\left(-\frac{\beta_r\rho^2}{2}\right)\rho\mathrm{d}\rho\right)\mathrm{d}\psi\\ &= \frac{\pi(\beta_x^2+\beta_y^2)}{\beta_r^2}\,,\quad \boldsymbol{\beta} = \beta_r\mathbf{e}_r + \beta_x\mathbf{e}_x + \beta_y\mathbf{e}_y \in \mathfrak{e}(2,\mathbb{R})\,.\end{aligned}$$

The expression of the probability density $\rho_{\boldsymbol{\beta}}$ of the Gibbs state indexed by $\boldsymbol{\beta} \in \mathfrak{e}(2,\mathbb{R})$, with respect to the Liouville measure $\mathrm{d}u\mathrm{d}v$, is

$$\rho_{\boldsymbol{\beta}}(u,v) = \exp\left(\frac{\beta_x^2+\beta_y^2}{2\beta_r}\right)\exp\left(-\frac{\beta_r({u'}^2+{v'}^2)}{2}\right)\,.$$

The expressions of the thermodynamic functions $E_J(\boldsymbol{\beta})$ (mean value of the moment map) and S (entropy) are

$$\begin{aligned}E_J(\boldsymbol{\beta}) &= \frac{2}{\beta_r}\boldsymbol{\varepsilon}_r - \frac{2\beta_x}{\beta_x^2+\beta_y^2}\boldsymbol{\varepsilon}_x - \frac{2\beta_y}{\beta_x^2+\beta_y^2}\boldsymbol{\varepsilon}_y\,,\\ S(\boldsymbol{\beta}) &= \log P(\boldsymbol{\beta}) = \log\pi + \log(\beta_x^2+\beta_y^2) - \log(\beta_r^2)\,.\end{aligned}$$

The expressions of the symplectic Lie group cocycle $\theta : \mathrm{E}(2,\mathbb{R}) \to \mathfrak{e}(2,\mathbb{R})^*$ and of the symplectic Lie algebra cocycle $\Theta : \mathfrak{e}(2,\mathbb{R})\times\mathfrak{e}(2,\mathbb{R}) \to \mathbb{R}$ associated to the moment map $J_{\mathbb{R}^2}$ (see for exemple [5]) can be determined by using the formulae

$$\theta(g) = J\circ\Phi_g - \mathrm{Ad}^*_{g^{-1}}\circ J\,,\quad \Theta(X,Y) = \left\langle T_e\theta(X),Y\right\rangle\,,$$

where $g \in \mathrm{E}(2,\mathbb{R})$, X and $Y \in \mathfrak{e}(2,\mathbb{R})$, $\Phi_g : \mathbb{R}^2 \to \mathbb{R}^2$ being the affine isometry of $\mathbb{R}^2$ determined by the action of $g \in \mathrm{E}(2,\mathbb{R})$. Although they are not necessary for the determination of Gibbs states, they are indicated below.

$$\theta(g_{(\varphi,x,y)}) = \frac{x^2+y^2}{2}\boldsymbol{\varepsilon}_r - y\boldsymbol{\varepsilon}_x + x\boldsymbol{\varepsilon}_y\,,$$
$$\Theta(r_1\mathbf{e}_r + x_1\mathbf{e}_x + y_1\mathbf{e}_y, r_2\mathbf{e}_r + x_2\mathbf{e}_x + y_2\mathbf{e}_y) = x_1y_2 - y_1x_2\,.$$

Remark 14 The fact that generalized temperatures are elements of the Lie algebra $\mathfrak{e}(2,\mathbb{R})$ whose component β_r on $\mathbf{e}_r$ is stricly positive may seem surprising, since there is apparently no reason explaining why clockwise and counter-clockwise rotations have different properties. I believe that it follows from the choice of $\mathrm{d}u\wedge\mathrm{d}v$ as a symplectic form on the plane $\mathbb{R}^2$, endowed with coordinates u and v. This choice automatically implies the choice of an orientation of this plane: the Hamiltonian vector field which admits $(u^2+v^2)/2$ as Hamiltonian is indeed the infinitesimal generator of counter-clockwise rotations around the origin. Replacing $\mathrm{d}u\wedge\mathrm{d}v$ by its opposite $\mathrm{d}v \wedge \mathrm{d}u$ would have as consequence the replacement of this vector field by its opposite, which is the infinitesimal generator of clockwise rotations.

4. Final comments

We have given a few examples of Gibbs states for the Hamiltonian action of a non-commutative Lie group on a symplectic manifold, even when the considered symplectic manifold is non-compact. However, we encoutered too several examples in which no Gibbs state can exist, the set of generalized temperatures being empty. All our examples are relative to two-dimensional symplectic manifolds. It seems interesting to look now at higher-dimensional symplectic manifolds. One may think that obstructions which occur for the existence of Gibbs states will increase with the dimension of the considered symplectic manifold, as well as with the dimension of its group of symmetries.

The Galilei group, so named in honour of the Italian scientist Galileo Galilei (1564–1642), is the group of symmetries of the mathematical model of space-time used in classical (non relativistic) Mechanics. It is a ten-dimensional Lie group diffeomorphic to $\mathrm{SO}(3) \times \mathbb{R}^7$. The Lie group of symmetries of any isolated classsical (non-relativistic) mechanical system must contain the Galilei group as a Lie subgroup. In his book [13], Souriau has proven that for any mechanical system made of a set of material objects whose total mass is non-zero, the set of generalized temperatures, for the action of the Galilei group on the manifold of motions, is empty. Therefore no Gibbs state can exist for these systems. However, Gibbs states for subgroups of the Galilei group do exist, an Souriau in his book has shown that they have interesting interpretations in physics and in cosmology.

The interested reader will find more results abut the Galilei group and its central extension, Bargmann's group, in [10–12].

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