From Statics to Dynamics: Equations which govern Equilibria and Motions of mechanical systems

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I. Introduction.

More than two hundred years before J.C., *Archimedes* undestood the basic principles of *Statics*. The mathematical formulation of the laws of *Dynamics* was developed much later, during the XVI-th, XVII-th and XVIII-th centuries, and reached a state of maturity at the end of the XIX-th century.

New views about *Space* and *Time* appeared at the beginning of the XX-th century, with the *Special and General Relativity theories*. Their integration in the mathematical description of the motion of mechanical systems was surprisingly easy, *but at a price* : the introduction of the concept of *Field*, made essential by the fact that *actions at a distance between material objects are no more admitted in Relativity theories*.

I. Introduction (2)

In this lecture I will present the main ideas which allowed the transition from Statics to Dynamics and the development of a usable mathematical formulation of the motion of mechanical systems. *Newton's laws*, *d'Alembert's Principle*, the method of *Virtual Work*, the *Lagrange differential*, *Lagrangian and Hamiltonian formulations of Dynamics* will be discussed.

The geometric formulation of relations between the Lagrangian and the Hamiltonian formalisms, due to *W.M. Tulczyjew*, explained in his nice little book *"Geometric formulation of physical theories"* [10] and in some related publications [8,9] will be presented.

II. Statics. 1. What is Statics ?

Statics is the study of *equilibria* of a material system, with respect to a given *reference frame*. The material system can be made of a *continuous medium* (a fluid or a more general deformable medium), or of an assembly of *several parts*, of which each may act on the other parts either by *contact*, or by *remote actions* (by means of gravitational, electrostatic or magnetic forces). *External objects*, which are not parts of the system, may also act on the system by contact or remote actions.

II. Statics. 2. The principles of Statics

The laws of Statics rest on two principles :

- the *principle of equality of action and reaction* : if a part A of a material system exerts on another part B of the system a "force" F, the part B exerts on A the opposite "force" $-F$;
- the *principle of vanishing of the total "force"* : when a system is in equilibrium, the sum of all "forces" which act on it vanishes.

This principle can be applied to the *whole system*, and to *each of its parts*, since when the system is in equilibrium, each of its parts also is in equilibrium ; for a continuous medium, it can be applied to *infinitesimal parts* of the medium.

Of course when this principle is applied to *some part* of a system, one must take into account all the "forces" which are exerted on that part, by *other parts of the system* as well as by *external objects*.

But what exactly is a *“Force”*?
The simplest mathematical representation of a “force” acting on a material object A set in the physical space $\mathcal{E}$ is a vector attached to a point P of A; in other words an element $\overrightarrow{F} \in T_P\mathcal{E}$. The point P is the *application point* of the force. Such a force tends to displace the application point P, by a *translation, in the direction of the vector $\overrightarrow{F}$*.

Another kind of “force” is called *couple* (or *pure moment*). It is the limit of a pair opposite forces mathematically represented by the vectors $\overrightarrow{F}(\varepsilon)$ and $-\overrightarrow{F}(\varepsilon)$, applied to points $P + \varepsilon\overrightarrow{k}$ and $P - \varepsilon\overrightarrow{k}$, the dependence on ε of $\overrightarrow{F}(\varepsilon)$ being such that the total momentum $2\varepsilon\overrightarrow{k} \times \overrightarrow{F}(\varepsilon)$ has a finite limit $\overrightarrow{\mathcal{M}}$ when $\varepsilon \to 0$. Such a couple tends to *rotate the material element at point P around an axis of rotation parallel to $\overrightarrow{\mathcal{M}}$*.

II. Statics. 4. What is a virtual work in Statics ?

A more general mathematical representation of forces in Statics uses the concepts of *virtual infinitesimal displacement* and *virtual infinitesimal work* (for a given virtual infinitesimal displacement).

A *virtual infinitesimal displacement* of a material object A set in the physical space $\mathcal{E}$ is a vector field $\overrightarrow{V}$ defined on A. The physical meaning of $\overrightarrow{V}$ is that one tries to apply to each point $P \in A$ an infinitesimal displacement proportional to $\overrightarrow{V}(P)$.

The "forces"applied to the material object A are mathematically described by a real valued function $\mathcal{W}_A$ defined on the set of vector fields on A, verifying

$$\mathcal{W}_A(\overrightarrow{V}) = 0 \quad \text{when} \quad \overrightarrow{V} = 0\,.$$

$\mathcal{W}_A(\overrightarrow{V})$ is the *virtual infinitesimal work* done by the forces applied to A for the virtual infinitesimal displacement A.

II. Statics. 4. What is a virtual work in Statics ? (2)

The set of all vector fields on A being very large, one generally considers only virtual infinitesimal displacements which belong to a *finite-dimensional subset* of the set of all vector fields. The choice of this subset is guided by physical considerations.

For example, if A is a rigid body, one often uses vector fields on A which belong to the *Lie algebra of infinitesimal Euclidean displacements* of A.

The choice of the function $\mathcal{W}_A$ is guided by physical considerations. The simplest choice is a *linear function* : with such a choice, the "forces" applied to the material element A are mathematically described by an element of the *dual space* of the space of infinitesimal displacements. Therefore, the "forces" applied to a rigid body are usually described by an element (sometimes called *torsor*) of the *dual space of the Lie algebra of infinitesimal Euclidean displacements*.

Remarks

1. Infinitesimal Euclidean displacements are used as infinitesimal virtual displacements *not only for solids*, because if one assumes that the forces *internal to the material element* A only depend on the distances between its internal parts, the virtual infinitesimal work made by these internal forces vanishes when the infinitesimal virtual displacement preserves distances.

2. For material elements with an *internal structure* (for example *magnetic materials*, or *liquid crystals*) fields on A *more general* than vector fields can be used as virtual infinitesimal displacements (see for example the books by *Darryl Holm*) [1].

3. Several authors, for example *Wlodzimierz Tulczyjev* [10], have used functions *more general* than linear functions for the mathematical description of the *virtual infinitesimal work* of forces.

II. Statics. 4. The method of virtual works in Statics

According to the *second principle of Statics*, when a material system is in equilibrium the total forces which act on it, and on each of its parts, vanishes.

Since $\mathcal{W}_A(\overrightarrow{V}) = 0$ when $\overrightarrow{V} = 0$, when a part A of the material system is in equilibrium, the virtual infinitesimal work $\mathcal{W}_A(\overrightarrow{V})$ of forces exerted on A vanishes for all its possible virtual infinitesimal displacements $\overrightarrow{V}$. Using this property is called the *method of virtual works* in Statics.

A suitable choice of the space of virtual infinitesimal displacements often allows *important simplifications* : for example when the virtual infinitesimal displacements used are infinitesimal Euclidean displacements, the virtual infinitesimal work of *internal forces* is zero, so one has to calculate only the virtual infinitesimal work of *external forces*, exerted on A by *other parts of the system* or by *external objects*.

III. Dynamics. 1. Newtonian Dynamics

Dynamics is the study of motions of a material system.

Classical, or *Newtonian* (*i.e.* non relativistic) *Dynamics* rests of the law, formulated by *Isaac Newton* in his famous book *Philosophiae naturalis principia mathematica* [6], wich states that when a force $\overrightarrow{F}$ acts on a material point, the *acceleration* $\overrightarrow{\gamma}$ of this material point is proportional to $\overrightarrow{F}$, the coefficient of proportionality m being the *mass* of the material point :

$$\overrightarrow{F} = m\overrightarrow{\gamma}\,.$$

With this law and the *law of gravitational interaction* (also formulated in his book), accoding to which the gravitational force exerted on a material point M, of mass m, by another material point M' of mass m' is directed towards M' and proportional to $mm'\bigl(d(M,M')\bigr)^{-2}$, Newton was able to explain the motions of planets in the Solar system (previously discovered by *Johannes Kepler*).

III. Dynamics. 2. D'Alembert's principle

Let us consider a material system which moves in the physical space $\mathcal{E}$. *Newton's law* states that each elementary part of the system, of mass m, on which, at time t, the total force exerted by other parts of the system and by external objects is $\overrightarrow{F}(t)$, is accelerated, with an acceleration $\overrightarrow{\gamma}(t)$ satisfying

$$\overrightarrow{F}(t) = m\overrightarrow{\gamma}(t)\,.$$

D'Alembert's principle is a way to reduce *problem in Dynamics* to an *equivalent problem in Statics*. It says that $\overrightarrow{F}_{\text{fictitious}}(t) = -m\overrightarrow{\gamma}(t)$ is a *fictitious force* exerted, at time t, on the elementary mass m when it is accelerated at an acceleration $\overrightarrow{\gamma}(t)$, and that the motion of this mass element is such that the *total force which acts on it, real $\overrightarrow{F}(t)$ plus fictitious $\overrightarrow{F}_{\text{fictitious}}(t)$*, vanishes identically at each time t :

$$\overrightarrow{F}(t) + \overrightarrow{F}_{\text{fictitious}}(t) = 0\,, \quad \text{with} \quad \overrightarrow{F}_{\text{fictitious}}(t) = -m\overrightarrow{\gamma}(t)\,.$$

Since *d'Alembert's principle* allows to reduce any problem in *Dynamics* to an equivalent problem in *Statics*, the *method of virtual works* can be used in Dynamics as well as in Statics. The method often offers a very convenient way for the derivation of the equations of motion of a mechanical system.

The method consists in writing that the motion of every part A of the material system is such that at any time, the virtual infinitesimal work of *all the forces (real and fictitious)* applied to A vanishes, for any virtual infinitesimal displacement of A.

Of course, the virtual infinitesimal displacements considered affect only the position of the various parts of A in Space, at a *given fixed time*.

III. Dynamics. 4. Lagrange dynamics.

In his famous book *Mécanique analytique* [3], Lagrange uses an $n+1$-dimensional manifold $\widetilde{Q}$ as *configuration space-time*; a surjective submersion $\theta : \widetilde{Q} \to \mathcal{T}$ maps $\widetilde{Q}$ onto the interval $\mathcal{T}$ of possible values of the time. In practice, when an origin and a unit of time are chosen, $\mathcal{T}$ is identified with an interval of the real line $\mathbb{R}$. Each $t \in \mathcal{T}$ is called a *time*, and the n-dimensional manifold $Q_t = \theta^{-1}(t)$ is the set of possible configurations of the system at time t. In local coordinates adapted to the submersion $\theta : \widetilde{Q} \to \mathcal{T}$

$$\widetilde{q} = (t, q^1, \ldots, q^n)\,, \quad \theta : (t, q^1, \ldots, q^n) \mapsto t\,.$$

A *motion* of the system is a smooth section $c : \mathcal{T} \to \widetilde{Q}$ of the submersion θ. In local coordinates

$$t \mapsto c(t) = \bigl(t, q^1(t), \ldots, q^n(t)\bigr)\,.$$

III. Dynamics. 4. Lagrange dynamics (2).

Assuming that a unit of length has been chosen, the physical space $\mathcal{E}$ is identified with a 3-dimensional affine Euclidean space. For each material element α of the system, of mass m_α (a positive number, when a unit of mass has been chosen), there is a smooth map $M_\alpha : \widetilde{Q} \to \mathcal{E}$, whose image $M_\alpha(\widetilde{q})$ is the position occupied in Space by the material element α when the time and the configuration of the mechanical system are mathematically described by the element $\widetilde{q} \in \widetilde{Q}$. Following Lagrange, we will first consider a particular material element α. At the end of the calculation we will make the sum over all the material elements of the system.

For a motion $t \mapsto c(t)$ of the system, the velocity and the momentum of the material element α are

$$\overrightarrow{V}_\alpha(t) = \frac{\overrightarrow{dM_\alpha} \circ c(t)}{dt}\,, \qquad \overrightarrow{p}_\alpha(t) = m_\alpha \overrightarrow{V}_\alpha(t)\,.$$

III. Dynamics. 4. Lagrange dynamics (3).

Lagrange writes the fundamental law of dynamics for the material element α

$$\frac{d\overrightarrow{p}_\alpha(t)}{dt} = \overrightarrow{F}_\alpha$$

where $\overrightarrow{F}_\alpha$ is the total force exerted on the material element α.

Remarks When writing this equality, Lagrange, following Newton, *implicitly makes an assumption on the structure of the physical Space* $\mathcal{E}$: the first and the second derivatives of $M_\alpha \circ c(t)$ with respect to the time t *are elements of different spaces* : $T_{M_\alpha \circ c(t)}\mathcal{E}$ and of $T_{\overrightarrow{V}(t)}(T\mathcal{E})$, respectively. It is the *triviality of the tangent bundle* $T\mathcal{E}$ which allows to consider them as elements of the associated Euclidean vector space $\overrightarrow{\mathcal{E}}$.

III. Dynamics. 4. Lagrange dynamics (4).

The force $\overrightarrow{F}_\alpha$ is an element of the cotangent space $T^*_{M_\alpha \circ c(t)}\mathcal{E}$, identified with $\overrightarrow{\mathcal{E}}^*$ by trivialization of the cotangent bundle. The *Euclidean scalar product allows its identification with* $\overrightarrow{\mathcal{E}}$.

By assuming the existence of the submersion $\theta : \widetilde{Q} \to \mathcal{T}$, Lagrange, following Newton, *assumes that there exists an absolute time*, the same for all parts of the mechanical system.

Then Lagrange uses the *principle of virtual work* : he considers an *infinitesimal virtual displacement* of the mechanical system and calculates the *infinitesimal virtual work* made by the time derivative $\dfrac{d\overrightarrow{p}_\alpha(t)}{dt}$ of the momentum $\overrightarrow{p}_\alpha(t)$ of the material element α, and by the force $\overrightarrow{F}_\alpha$ exerted on that element. And he writes the equality of these virtual infinitesimal works.

III. Dynamics. 4. Lagrange dynamics (5).

Following Lagrange, we will denote by δq the virtual infinitesimal displacement, although this notation is misleading : *it is not a differential form*, but rather a *vector field* tangent to the configuration space-time $\widetilde{Q}$ along the the curve $\{c(t); t \in \mathcal{T}\}$.

Moreover, its projection onto $\mathcal{T}$ must vanish : for each $t \in \mathcal{T}$, we must have

$$T_{c(t)}\theta\Bigl(\delta q\bigl(c(t)\bigr)\Bigr) = 0\,.$$

This condition expresses the fact that at each time t, the virtual infinitesimal displacement only affects the configuration of the system, not the time t.

The tangent bundle $T\mathcal{E}$ being trivial, we identify it with $\mathcal{E} \times \overrightarrow{\mathcal{E}}$ and we denote by $\mathrm{pr}_2 : T\mathcal{E} = \mathcal{E} \times \overrightarrow{\mathcal{E}} \to \overrightarrow{\mathcal{E}}$ the second projection. We set

$$\overrightarrow{Z}_\alpha = \mathrm{pr}_2 \circ TM_\alpha : T\widetilde{Q} \to \overrightarrow{\mathcal{E}}\,.$$

III. Dynamics. 5. The virtual work of accelerations.

The virtual infinitesimal work of $\dfrac{d\overrightarrow{p}_\alpha(t)}{dt}$ is

$$\mathcal{W}\left(\frac{d\overrightarrow{p}_\alpha(t)}{dt}\,,\;\delta q\right) = \left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\right\rangle\,.$$

The pairing $\langle\ ,\ \rangle$ on the left-hand side of this formula stands for the Euclidean scalar product of vectors in $\overrightarrow{\mathcal{E}}$.

The calculation made by Lagrange (presented in *Appendix A*) aims at expressing this infinitesimal virtual work as the pairing of the vector $\delta q\bigl(c(t)\bigr) \in T_{c(t)}\widetilde{Q}$ with a covector, element of $T^*_{c(t)}\widetilde{Q}$.

The result of is the following. Let $T_\alpha : T\widetilde{Q} \to \mathbb{R}$ be the function

$$T_\alpha = \frac{m_\alpha}{2}\,\langle\overrightarrow{Z}_\alpha, \overrightarrow{Z}_\alpha\rangle\,.$$

T_α is the *kinetic energy* of the material element α.

III. Dynamics. 5. The virtual work of accelerations (2).

Lagrange obtains the following expression for the virtual work of $\dfrac{dp_\alpha(t)}{dt}$:

$$\mathcal{W}\left(\frac{d\overrightarrow{p}_\alpha(t)}{dt}\,,\;\delta q\right) = \left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle$$

$$= \sum_{i=1}^{n} \left[\left(\frac{d}{dt} \left(\frac{\partial T_\alpha}{\partial \dot q^i} \circ \frac{dc(t)}{dt} \right) - \frac{\partial T_\alpha}{\partial q^i} \circ \frac{dc(t)}{dt} \right) \left(\delta q^i \circ c(t) \right) \right] .$$

This virtual work is expressed as the pairing of the vector $\delta q \circ c(t) \in T_{c(t)}\widetilde{Q}$ with a covector, element of $T^*_{c(t)}\widetilde{Q}$. More exactly, since $\delta q \circ c(t) \in \ker T_{c(t)}\theta$, that covector is determined only up to addition of any covector which vanishes on $\ker T_{c(t)}\theta$; in other words it is an element of the quotient space $T^*_{c(t)}\widetilde{Q}/\bigl(\ker T_{c(t)}\theta\bigr)$.

Following Lagrange, we now sum over all material elements α of the system. The sum of all the virtual infinitesimal works

$$\mathcal{W}_{\mathrm{acc}}(\delta q) = \sum_{\alpha} \mathcal{W}\left(\frac{d\overrightarrow{p}_{\alpha}(t)}{dt}\,,\ \delta q\right)$$

will be called the *virtual infinitesimal work of acceleration quantities* of the system, for the virtual infinitesimal displacement δq. The real valued function (defined on the subset of $T\widetilde{Q}$ made by vectors wose projection on the time axis $\mathcal{T}$ is equal to 1)

$$T = \sum_{\alpha} T_{\alpha}$$

is such that $T \circ \dfrac{dc(t)}{dt}$ is the *total kinetic energy of the system* when its motion is $t \mapsto c(t)$.

III. Dynamics. 5. The virtual work of accelerations (4).

When the system is made by a finite number of material points, the sums over all values of α are finite. In other cases these sums should be replaced by integrals.

Finally Lagrange obtains for the *virtual infinitesimal work of acceleration quantities* of the system

$$\mathcal{W}_{\mathrm{acc}}(\delta q)$$
$$= \sum_{i=1}^{n} \left[\left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \right) - \frac{\partial T}{\partial q^i} \circ \frac{dc(t)}{dt} \right) \left(\delta q^i \circ c(t) \right) \right] .$$

This virtual work is expressed as the pairing of the vector $\delta q \circ c(t) \in T_{c(t)}\widetilde{Q}$ with a covector, element of $T^*_{c(t)}\widetilde{Q}$. More exactly, since $\delta q \circ c(t) \in \ker T_{c(t)}\theta$, that covector is determined only up to addition of any covector which vanishes on $\ker T_{c(t)}\theta$; in other words it is an element of the quotient space $T^*_{c(t)}\widetilde{Q}/\bigl(\ker T_{c(t)}\theta\bigr)$.

III. Dynamics. 6. The virtual work of forces.

The virtual work of the force $\overrightarrow{F}_\alpha$ exerted on the material element α

$$\mathcal{W}(\overrightarrow{F}_\alpha\,,\ \delta q) = \bigl\langle \overrightarrow{F}_\alpha, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \bigr\rangle$$

can be expressed in terms of the pull-back $\Psi_\alpha = M_\alpha^*(\overrightarrow{F}_\alpha)$ of $\overrightarrow{F}_\alpha$ (considered as a covector, element of $T^*_{M_\alpha \circ c(t)}\mathcal{E}$) by the map $M_\alpha : \widetilde{Q} \to \mathcal{E}$. We may write

$$\mathcal{W}(\overrightarrow{F}_\alpha\,,\ \delta q) = \Bigl\langle \Psi_\alpha\bigl(c(t)\bigr), \delta q \circ c(t) \Bigr\rangle\,.$$

Summing over all the material elements α, we obtain the *virtual work of forces* acting on all material elements of the system

$$\mathcal{W}_{\text{forces}}(\delta q) = \Bigl\langle \Psi\bigl(c(t)\bigr), \delta q \circ c(t) \Bigr\rangle\,, \quad \text{with} \quad \Psi = \sum_\alpha \Psi_\alpha\,.$$

By writing $\Psi\bigl(c(t)\bigr)$, we assumed that the applied forces only depend on the configuration of the system and on the time ; under this assumption, Ψ is a differential 1-form on the configuration space-time $\widetilde{Q}$ of the system (defined up to addition of a form which vanishes on $\ker T\theta$; in other words, Ψ is a smooth section of the bundle $\bigl(T^*\widetilde{Q}/(\ker T\theta)^0\bigr) \to \widetilde{Q}$). More generally, if there are forces depending on the velocities of some parts of the system, Ψ is a semi-basic 1-form on $T\widetilde{Q}$.

The mechanical system's equations of motion are obtained by writing that, for any virtual infinitesimal displacement δq,

$$\mathcal{W}_{\text{acc}}(\delta q) = \mathcal{W}_{\text{forces}}(\delta q)\,, \quad \text{or in local coordinates,}$$

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q^i}\circ\frac{dc(t)}{dt}\right) - \frac{\partial T}{\partial q^i}\circ\frac{dc(t)}{dt}\right)\bigl(\delta q^i\circ c(t)\bigr) = \Psi_i\circ c(t)\,.$$

III. Dynamics. 7. The Lagrange equations (2)

The applied forces are said to be *conservative* when there exists a smooth function $\Phi : \widetilde{Q} \to \mathbb{R}$ such that

$$\Psi_i = \frac{\partial \Phi}{\partial q^i}\,.$$

The equations of motion then take the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt}\right) - \frac{\partial T}{\partial q^i} \circ \frac{dc(t)}{dt} = \frac{\partial \Phi}{\partial q^i} \circ c(t)\,, \quad \text{or}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt}\right) - \frac{\partial L}{\partial q^i} \circ \frac{dc(t)}{dt} = 0\,, \quad \text{with}$$

$$L \circ \frac{dc(t)}{dt} = T \circ \frac{dc(t)}{dt} + \Phi \circ c(t)\,.$$

III. Dynamics. 7. The Lagrange equations (3)

The real-valued function L, defined on the subset of $T\widetilde{Q}$ of vectors whose projection on the time axis $\mathcal{T}$ is equal to 1, is called the *Lagrangian*, and the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\circ\frac{dc(t)}{dt}\right)-\frac{\partial L}{\partial q^i}\circ\frac{dc(t)}{dt}=0$$

are the famous *Lagrange equations*.

In local coordinates $(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ they have the well known expression

$$\begin{aligned}\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\left(t, q^1(t), \ldots, q^n(t), \frac{dq^1(t)}{dt}, \ldots, \frac{dq^n(t)}{dt}\right)\right]\\ -\frac{\partial L}{\partial q^i}\left(t, q^1(t), \ldots, q^n(t), \frac{dq^1(t)}{dt}, \ldots, \frac{dq^n(t)}{dt}\right)=0\,.\end{aligned}$$

III. Dynamics. 8. The Lagrange differential

In the Lagrange equations of our mechanical system, le Lagrangian L is the sum of the kinetic energy T (function defined on the subset $T^1\widetilde{Q}$ of $T\widetilde{Q}$ of vectors whose projection on the time axis $\mathcal{T}$ is equal to 1) and of a potential Φ (defined on $\widetilde{Q}$) composed with the projection $\tau_{\widetilde{Q}} : T\widetilde{Q} \to \widetilde{Q}$.

However, Lagrange equations can be written with any smooth function L defined on $T^1\widetilde{Q}$ as Lagrangian. For a given smooth section c of $\theta : \widetilde{Q} \to \mathcal{T}$ and a given time $t \in \mathcal{T}$, the left hand side of the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt}\right) - \frac{\partial L}{\partial q^i} \circ \frac{dc(t)}{dt}$$

only depends of the 2-jet $j^2c(t)$ of the section c at point t, and takes its values in the quotient space $T^*_{c(t)}\widetilde{Q}/(\ker T_{c(t)}\theta)^0$.

III. Dynamics. 8. The Lagrange differential (2)

Therefore, the Lagrangian L determines a smooth bundle map

$$\Delta_L : J^2\bigl(\Gamma(\theta)\bigr) \to T^*\widetilde{Q}/(\ker T\theta)^0$$

called the *Lagrange differential* of L, defined on the space $J^2\bigl(\Gamma(\theta)\bigr)$ of 2-jets of sections of the projection $\theta : \widetilde{Q} \to \mathcal{T}$, with values in the quotient $T^*\widetilde{Q}/(\ker T\theta)^0$ of the cotangent bundle $T^*\widetilde{Q}$ by the rank 1 bundle of covectors which vanish on $\ker T\theta$.

$T^*\widetilde{Q}/(\ker T\theta)^0$ is a *Poisson manifold* since it is the quotient of the symplectic manifold $(T^*\widetilde{Q}, \omega_{\widetilde{Q}})$ by a foliation whose leaves are 1-dimensional, hence isotropic. Its *symplectic leaves* are its submanifolds on which the time function θ (composed with the projection onto $\widetilde{Q}$) is constant. For each $t \in \mathcal{T}$, the symplectic leaf which projects on t is symplectically diffeomorphic to the cotangent bundle T^*Q_t, with $Q_t = \theta^{-1}(t)$. This leaf is the *phase space* of the system at time t.

W.M. Tulczyjev [8] has shown that the *Lagrange differential* is part of a complex, the *Lagrange complex*, which plays an inportant part in the inverse problem of calculus of variations.

For each smooth section $c : [t_0, t_1] \to \widetilde{Q}$ of the projection $\theta : \widetilde{Q}$, the *action integral* is

$$S(c) = \int_{t_0}^{t_1} L \circ \frac{dc(t)}{dt}\, dt\,.$$

The famous Irish scientist *William Rowan Hamilton* [2] has shown that the *variation* of $S(c)$ for an infinitesimal variation δc of c which leaves fixed the end points $\bigl(t_0, c(t_0)\bigr)$ and $\bigl(t_1, c(t_1)\bigr)$, vanishes if and only if $\Delta_L\bigl(j^2c(t)\bigr) = 0$ for all $t \in [t_0, t_1]$. We have

$$\delta S(c, \delta c) = \int_{t_0}^{t_1} \Bigl\langle \Delta_L\bigl(j^2c(t)\bigr), \delta c(t)\Bigr\rangle\, dt\,.$$

The pairing $\langle\ ,\ \rangle$ in the right hand side is the pairing of the equivalence class of covectors $\Delta_L\bigl(j^2c(t)\bigr) \in T^*_{c(t)}\widetilde{Q}/(\ker T_{c(t)}\theta)^0$ with the vector $\delta c(t) \in \ker T_{c(t)}\theta \subset T_{c(t)}\widetilde{Q}$.

III. Dynamics. 10. The energy function

Let us define on $T^1\widetilde{Q}$ the *energy function*

$$E(t,q,\dot{q}) = \sum_{i=1}^{n} \dot{q}^i \frac{\partial L(t,q,\dot{q})}{\partial \dot{q}^i} - L(t,q,\dot{q})\,.$$

and the 1-form

$$\sigma = \sum_{i=1}^{n} \frac{\partial L(t,q,\dot{q})}{d\dot{q}^i}\, dq^i - E(t,q,\dot{q})\, dt\,,$$

For any smooth section $c : [t_0, t_1] \to \widetilde{Q}$ of θ, the action integral can be expressed as (the proof is presented in *Appendix A*)

$$S(c) = \int \left(\frac{dc(t)}{dt}\right)^* \sigma\,.$$

We recall that a smooth section $c : [t_0, t_1] \to \widetilde{Q}$ of θ satisfies the *principle of virtual work* if and only if the action integral $S(c)$ is stationary for the intinitesimal variations of c with fixed endpoints.

Using the above expression of the action integral, on can prove that $c : [t_0, t_1] \to \widetilde{Q}$ of θ satisfies the *principle of virtual work* if and only if, for each $t \in]t_0, t_1[$,

$$i\left(\frac{d^2c(t)}{dt^2}\right) d\sigma = 0\,.$$

This equation is the *intrisic form of the Lagrange equations*.

III. Dynamics. 12. The Legendre map $\mathcal{L}_L$

The Legendre map $\mathcal{L}_L$, expressed in local coordinates $(t, q^i, \dot{q}^i)$ on $T^1\widetilde{Q}$ (submanifold of $T\widetilde{Q}$ on which $\dot{t} = 1$) is

$$\mathcal{L}_L : (t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_i = \frac{\partial L(t, q^i, \dot{q}^i)}{\partial \dot{q}^i} \right) .$$

It is defined on $T^1\widetilde{Q}$, and takes its values in the quotient bundle $T^*\widetilde{Q}/(\ker T\theta)^0$.

The Lagrangian L is said to be *regular* when the Legendre map $\mathcal{L}_L$ is everywhere of rank $2n+1$, and *hyperregular* when $\mathcal{L}_L$ is a diffeomorphism.

When L is regular, $d\sigma$ is of rank $2n$ (see the proof in Appendix A), and there exists on $T^1\widetilde{Q}$ a unique vector field $\mathcal{X}_L$ contained in $\ker d\sigma$ whose projection on $\mathcal{T}$ is equal to 1. Integral curves of this vector field are *motions of the mechanical system*.

Still when L is regular, the *manifold of motions* of the mechanical system is the quotient of the presymplectic manifold $(T^1\widetilde{Q}, d\sigma)$ by its characteristic foliation determined by $\ker d\sigma$.
J. M. Souriau [7] has shown that it has indeed the structure of a *smooth symplectic manifold* (maybe non-Hausdorff).

III. Dynamics. 14. The Hamiltonian formalism

We now assume that $\widetilde{Q} = \mathcal{T} \times Q$, where $\mathcal{T}$ is the time axis and Q a *configuration manifold*. The map $\theta : \widetilde{Q} \to \mathcal{T}$ is the first projection. The codimension 1 submanifold $T^1\widetilde{Q}$ can be identified with $\mathcal{T} \times TQ$, and the quotient manifold $T^*\widetilde{Q}/(\ker T\theta)^0$ with $\mathcal{T} \times T^*Q$. The *Legendre map* determined by the Lagrangian L can therefore be considered as a map $\mathcal{L}_L : \mathcal{T} \times TQ \to \mathcal{T} \times T^*Q$,

$$\mathcal{L}_L : (t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_i = \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \right), \quad 1 \leq i \leq n\,.$$

The cotangent bundle $T^*\widetilde{Q}$ can be identified with $T^*\mathcal{T} \times T^*Q$. We define the map

$$\widehat{\mathcal{L}}_L : \mathcal{T} \times TQ \to T^*\mathcal{T} \times T^*Q\,, \quad \widehat{\mathcal{L}}_L = \mathcal{L}_L - E\,dt\,.$$

We still assume that $\widetilde{Q} = \mathcal{T} \times Q$ and, in addition, that the Lagrangian L is *hyperregular*. The *Hamiltonian* is the function

$$H = E \circ \mathcal{L}_L^{-1} : \mathcal{T} \times T^*Q \to \mathbb{R}\,.$$

We have seen that the *motions* of the mechanical system are integral curves of a vector field $\mathcal{X}_L$, defined on $T^1\widetilde{Q} = \mathcal{T} \times TQ$, such that

$$i(\mathcal{X}_L)d\sigma = 0\,, \quad T\theta(\mathcal{X}_L) = 1\,.$$

The image $W = \widehat{\mathcal{L}}_L(\mathcal{T} \times TQ)$ of the map $\widehat{\mathcal{L}}_L$ is a submanifold of $T^*\mathcal{T} \times T^*Q$, on which we can define the vector field, direct image of $\mathcal{X}_L$ by the map $\widehat{\mathcal{L}}_L : \mathcal{T} \times TQ \to W$,

$$\mathcal{Y}_L = (\widehat{\mathcal{L}}_L)_*(\mathcal{X}_L)\,.$$

The map

$$(t, q^i, p_i) \mapsto (t, q^i, p_t = H(t, q^i, p_i)), \quad 1 \leq i \leq n,$$

allows us to identify $\mathcal{T} \times T^*Q$ with the submanifold W of $T^*\widetilde{Q}$.
Using this identification of $\mathcal{T} \times T^*Q$ with W, the form induced on W by the Liouville 1-form of $T^*\widetilde{Q} = T^*\mathcal{T} \times T^*Q$ becomes

$$\eta_Q - H\,dt,$$

where η_Q is the Liouville 1-form on T^*Q
The vector field $\mathcal{Y}_L$, now considered as defined on $\mathcal{T} \times T^*Q$, is therefore determined by

$$i(\mathcal{Y}_L)(d\eta_Q - dH \wedge dt) = 0, \quad T\pi_{\mathcal{T}}(\mathcal{Y}_L) = 1.$$

The second equality above allows us to write

$$\mathcal{Y}_L = X_H + \frac{\partial}{\partial t}\,,$$

where X_H is a *time-dependent vector field* on T^*Q.
The first equality determining $\mathcal{Y}_L$ leads to

$$i(X_H)d\eta_Q = -\bigl(dH - \frac{\partial H}{\partial t}\,dt\bigr)\,,\quad i(X_H)dH = 0\,.$$

The first equation shows that for each fixed time t, the value X_{H_t} of the time-dependent vector field X_H is the Hamiltonian vector field on T^*Q whose Hamiltonian is $H_t : T^*Q \to \mathbb{R}$. The second equation is automatically satisfied when the first equation is satisfied.

This is the *Hamiltonian formalism*, equivalent to the Lagrangian formalism when the Lagrangian L is hyperregular.

IV. Relativistc Dynamics. 1. A new setting

In *Classical Dynamics*, Time is set apart from Space : the theory fundamentally depends on the concepts of *time ordering* and *simultaneity* of events which occur at different places in Space.

In *Relativistic Physics*, Time and Space are merged into a *single, structured Space-Time* ; the concepts of time ordering and simultaneity *are no more universally valid*.

Instantaneous action at a distance of a material objet A on another material object B is no more admitted : the new concept of *field* must be taken into account. Actions of a material object A on another, distant material object B only occur when *fields* are created (or modified) by A ; the newly created (of modified) fields propagate until they reach B, and then act on it.

A *complete theory of Relativistic Dynamics* in Space-Time should consider both *material objects* and *fields* and describe their *mutual interactions*.

However, *Newton's law*, *d'Alembert's principle* and the *method of virtual works* still can be used for the motion of a *point-like particle in Space-Time* : we only have to use the *inertial reference frame* in which the *particle is at rest*, at the event at which these laws are expressed.

For example, let $\mathcal{M}$ be the Minkowski space-time (it is an affine, pseudo-Euclidean 4-dimensional space, the associated vector space $\overrightarrow{\mathcal{M}}$ being endowed with a pseudo-Euclidean scalar product $(\ |\ )$ with signature $(+,-,-,-)$). The world line of a point particle M moving in $\mathcal{M}$ is a time-like curve $\mathcal{C}$, assumed to be smooth. We will parametrize $\mathcal{C}$ by the *proper time* of the particle : it is the *arc length* s along $\mathcal{C}$, measured from an origin event $M_0 = M(0)$. The unit vector $\overrightarrow{\dfrac{dM(s)}{ds}}$ tangent to $\mathcal{C}$ at the event $M(s)$ determines the inertial reference frame in which the particle is at rest at the event $M(s)$.

Newton's law is ($\overrightarrow{F}(s)$ being the *force*)

$$\overrightarrow{F}(s) = m\frac{\overrightarrow{d^2M(s)}}{ds^2}\,, \quad \text{with} \quad \left(\overrightarrow{F}(s)\;\middle|\;\frac{\overrightarrow{dM(s)}}{ds}\right) = 0\,.$$

A *virtual infinitesimal displacement* of the particle at the event $M(s)$ is a vector $\overrightarrow{w}$ tangent to $\mathcal{M}$ at the event $M(s)$, *space-like* with respect to the reference frame in which the particle is at rest at the event $M(s)$, *i.e.* orthogonal to $\dfrac{\overrightarrow{dM(s)}}{ds}$. The corresponding *infinitesimal virtual work* of the acceleration quantity of the particle is (the minus sign compensates the definite-negativeness of the scalar product of spacelike vectors)

$$-\left(m\frac{\overrightarrow{d^2M(s)}}{ds^2}\;\middle|\;\overrightarrow{w}\right)\,.$$

For an observer at rest with respect to an inertial reference frame, in which the coordinates are (t, x, y, z), the motion of the particle is described by the parametrized curve $t \mapsto M \circ s(t)$. The square v^2 of the velocity of the paticle with respect to the observer is

$$v^2 = \left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2 .$$

We set

$$v^2 = c^2 \tanh^2 \eta \, .$$

Using $c^2 ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$, we see that

$$\frac{ds}{dt} = \frac{1}{\cosh \eta} = \sqrt{1 - \frac{v^2}{c^2}} \, .$$

In terms of the coordinates in the observer's frame, Newton's law is

$$\frac{\overrightarrow{F}(t)}{\cosh\bigl(\eta(t)\bigr)} = \frac{d}{dt}\left(m\cosh\bigl(\eta(t)\bigr)\,\frac{\overrightarrow{d\bigl(M\circ s(t)\bigr)}}{dt}\right).$$

Let us choose the coordinate system in the reference frame of the observer so that for a given value t_0 of t,

$$\frac{dx(t)}{dt}\Bigl|_{t=t_0} = v\,,\qquad \frac{dy(t)}{dt}\Bigl|_{t=t_0} = 0\,,\qquad \frac{dz(t)}{dt}\Bigl|_{t=t_0} = 0\,.$$

The time component of Newton's equation is

$$\frac{F_0(t_0)}{\cosh\bigl(\eta(t_0)\bigr)} = \frac{d}{dt}\Bigl(m\cosh\bigl(\eta(t)\bigr)\Bigr)\Bigl|_{t=t_0}\ .$$

The three space components of Newton's equation are

$$\frac{F_x(t_0)}{\cosh\bigl(\eta(t_0)\bigr)} = \frac{d}{dt}\left(m\cosh\bigl(\eta(t)\bigr)\frac{dx(t)}{dt}\right)\Big|_{t=t_0},$$

$$\frac{F_y(t_0)}{\cosh\bigl(\eta(t_0)\bigr)} = \frac{d}{dt}\left(m\frac{dy(t)}{dt}\right)\Big|_{t=t_0},$$

$$\frac{F_z(t_0)}{\cosh\bigl(\eta(t_0)\bigr)} = \frac{d}{dt}\left(m\frac{dz(t)}{dt}\right)\Big|_{t=t_0}.$$

By analogy with the usual Newton's law, physicits interpret these formulae in terms of an *apparent mass* of the particle in the observer's reference frame. This apparent mass is $m\cosh\bigl(\eta(t)\bigr)$ for longitudinal forces (acting in the direction of the velocity v), and m for transverse forces.

Since in the Minkowski Space-Time $\mathcal{M}$ there is no privileged time, the action integral for a point-like particle should be invariant by any admissible change of parametrization of the particle's world line. The Lagrangin should therefore be a homogeneous function of degree 1 on the tangent bundle $T\mathcal{M}$. For a free particle, the action integral should be expressed in geometric, invariant terms. The most obvious expression is

$$\widehat{S}(\widehat{c}) = k \int_{s_0}^{s_1} \sqrt{\left(\frac{\overrightarrow{dM(s)}}{ds} \,\Big|\, \frac{\overrightarrow{dM(s)}}{ds} \right)} ds \,.$$

The constant k can be determined by looking at the classical limit :

$$k = -mc \,.$$

When the world line of the particle is parametrized by the time t relative to some inertial frame, the action integral becomes

$$\widehat{S}(\widehat{c}) = -mc \int_{t_0}^{t_1} \sqrt{c^2 - \bigl(v(t)\bigr)^2}\, dt\,.$$

When the relative velocity v of the particle in the considered reference frame is small, this action integral becomes approximately

$$\int_{t_0}^{t_1} m \left(-c^2 + \frac{\bigl(v(t)\bigr)^2}{2} \right) dt\,.$$

We recognize the opposite of the rest energy mc^2 of the particle, which plays no part in the search of extremals, plus its kinetic energy $\dfrac{m\bigl(v(t)\bigr)^2}{2}$ relative to the considered reference frame.

V. The Tulczyjew isomorphisms. 1. Canonical involution of $T(TQ)$.

I will now describe the very nice geometric interpretation of the relations between the Lagrangian and the Hamiltonian formalisms, due to *W.M. Tulczyjew*. Let Q be a smooth manifold (the *configuration manifold* of a mechanical system). The second tangent bundle $T(TQ)$ has two different vector bundle structures :

- the tangent bundle structure $\tau_{TQ} : T(TQ) \to TQ$,
- the prolongation to vectors $T\tau_Q : T(TQ) \to TQ$ of the vector bundle structure $\tau_Q : TQ \to Q$.

Local coordinates on Q, TQ and $T(TQ)$ will be denoted by (x^i), (x^i, v^j) and $(x^i, v^j, \dot{x}^k, \dot{v}^l)$ respectively $(1 \leq i, j, k, l \leq n)$.

We have the commutative diagram

$$
\begin{tikzcd}
T(TQ) \arrow[r, "T\tau_Q"] \arrow[d, "\tau_{TQ}"'] & TQ \arrow[d, "\tau_Q"] \\
TQ \arrow[r, "\tau_Q"] & Q
\end{tikzcd}
\qquad
\begin{tikzcd}
(x^i, v^j, \dot{x}^k, \dot{v}^l) \arrow[r, mapsto, "T\tau_Q"] \arrow[d, mapsto, "\tau_{TQ}"'] & (x^i, \dot{x}^k) \arrow[d, mapsto, "\tau_Q"] \\
(x^i, v^j) \arrow[r, mapsto, "\pi_N"] & (x^i)
\end{tikzcd}
$$

There exists a canonical involutive vector bundle isomorphism $\kappa_Q : T(TQ) \to T(TQ)$ which exchanges the two vector bundle structures of $T(TQ)$. Its expression in local coordinates is

$$\kappa_Q : (x^i, v^j, \dot{x}^k, \dot{v}^l) \mapsto (x^i, \dot{x}^k, v^j, \dot{v}^l)\,.$$

The isomorphism κ_Q makes the following diagram commutative :

$$
\begin{tikzcd}
T(TQ) \arrow[r, "\kappa_Q"] \arrow[d, "\tau_{TQ}"'] & T(TQ) \arrow[d, "T\tau_Q"] \\
TQ \arrow[r, leftrightarrow, "\mathrm{id}_{TQ}"] & TQ
\end{tikzcd}
\qquad
\begin{tikzcd}
(x^i, v^j, \dot{x}^k, \dot{v}^l) \arrow[r, mapsto, "\kappa_Q"] \arrow[d, mapsto, "\tau_{TQ}"'] & (x^i, \dot{x}^k, v^j, \dot{v}^l) \arrow[d, mapsto, "T\tau_Q"] \\
(x^i, v^j) \arrow[r, leftrightarrow, "\mathrm{id}_{TQ}"] & (x^i, v^j)
\end{tikzcd}
$$

V. The Tulczyjew isomorphisms. 2. The α_Q isomorphism.

W.M. Tulczyjew defined the vector bundle isomorphism α_Q as the *transpose* of $\kappa_Q : \bigl(T(TQ), \tau_{TQ}, TQ\bigr) \to \bigl(T(TQ), T\tau_Q, TQ\bigr)$.

The vector bundle $\bigl(T(TQ), \tau_{TQ}, TQ\bigr)$ is the tangent bundle to TQ. Its dual bundle is $\bigl(T^*(TQ), \pi_{TQ}, TQ\bigr)$, the cotangent bundle to TQ. In local coordinates, the pairing by duality is

$$\Bigl\langle (x^i, v^j, p_{x^k}, p_{v^l}), (x^i, v^j, \dot{x}^k, \dot{v}^l)\Bigr\rangle = \sum_{k=1}^{n} \langle p_{x^k}, \dot{x}^k\rangle + \sum_{l=1}^{n} \langle p_{x^l}, \dot{v}^l\rangle\,.$$

The dual of the vector bundle $\bigl(T(TQ), T\tau_Q, TQ\bigr)$ is $\bigl(T(T^*Q, T\pi_Q, TQ)\bigr)$. The pairing by duality is

$$\Bigl\langle (x^i, p_{x^k}, \dot{x}^j, \dot{p}_{x^l}), (x^i, v^l, \dot{x}^j, \dot{v}^k)\Bigr\rangle = \sum_{k=1}^{n} \langle p_{x^k}, \dot{v}^k\rangle + \sum_{l=1}^{n} \langle \dot{p}_{x^l}, v^l\rangle\,.$$

V. The Tulczyjew isomorphisms. 2. The α_Q isomorphism (2).

By writing that α_Q is the transpose of κ_Q, we easily obtain its expression in local coordinates :

$$
\begin{tikzcd}
T(T^*Q) \arrow[r, "\alpha_Q"] \arrow[d, "T\pi_Q"'] & T^*(TQ) \arrow[d, "\pi_{TQ}"] \\
TQ \arrow[r, leftrightarrow, "\mathrm{id}_{TQ}"] & TQ
\end{tikzcd}
\qquad
\begin{tikzcd}
(x^i, p_{x^k}, \dot{x}^j, \dot{p}_{x^l}) \arrow[r, mapsto, "\alpha_Q"] \arrow[d, mapsto, "T\pi_Q"'] & (x^i, \dot{x}^j, \dot{p}_{x^l}, p_{x^k}) \arrow[d, mapsto, "\pi_{TQ}"] \\
(x^i, \dot{x}^j) \arrow[r, leftrightarrow, "\mathrm{id}_{TQ}"] & (x^i, \dot{x}^j)
\end{tikzcd}
$$

V. The Tulczyjew isomorphisms. 3. The β_Q isomorphism.

The canonical symplectic 2-form of T^*Q determines, up to the choice of a sign, an isomorphism between the tangent and the cotangent bundles to T^*Q. Therefore, there exists another vector bundle isomorphism

$$\beta_Q : \big(T(T^*Q), \tau_{T^*Q}, T^*Q\big) \to \big(T^*(T^*Q), \pi_{T^*Q}, T^*Q\big) \,.$$

With a suitable choice of the sign of β_Q,

$${\alpha_Q}^* \omega_{TQ} = {\beta_Q}^* \omega_{T^*Q} \,,$$

where ω_{TQ} and ω_{T^*Q} are the canonical symplectic 2-forms on $T^*(TQ)$ and $T^*(TQ)$, respectively. The choice of sign of β_Q for which this equality holds is

$$\beta_Q : v \mapsto i(v)\omega_Q \,.$$

One easily obtains the β_Q isomorphism's expression in local coordinates :

$$
\begin{tikzcd}
T(T^*Q) \arrow[r, "\beta_Q"] \arrow[d, "\tau_{T^*Q}"'] & T^*(T^*Q) \arrow[d, "\pi_{*TQ}"] \\
T^*Q \arrow[r, leftrightarrow, "\mathrm{id}_{T^*Q}"] & TQ
\end{tikzcd}
\qquad
\begin{tikzcd}
(x^i, p_{x^k}, \dot{x}^j, \dot{p}_{x^l}) \arrow[r, mapsto, "\beta_Q"] \arrow[d, mapsto, "\tau_{T^*Q}"'] & (x^i, p_{x^k}, \dot{p}_{x^l}, -\dot{x}^j \arrow[d, mapsto, "\pi_{T^*Q}"] \\
(x^i, p_{x^k}) \arrow[r, leftrightarrow, "\mathrm{id}_{T^*Q}"] & (x^i, p_{x^k})
\end{tikzcd}
$$

$$
\begin{tikzcd}
T^*(T^*Q) \arrow[d, "\pi_{T^*Q}"'] & T(T^*Q) \arrow[l, "\beta_Q"'] \arrow[r, "\alpha_Q"] \arrow[dl, "\tau_{T^*Q}"] \arrow[dr, "T\pi_Q"] & T^*(TQ) \arrow[d, "\pi_{TQ}"] \\
T^*Q \arrow[r, "\pi_Q"'] & Q & TQ \arrow[l, "\tau_Q"]
\end{tikzcd}
$$

In local coordinates

$$
\begin{tikzcd}
(x^i, p_{x^k}, \dot p_{x^l}, -\dot x^j) \arrow[d, "\pi_{T^*Q}"'] & (x^i, p_{x^k}, \dot x^j, \dot p_{x^l}) \arrow[l, mapsto, "\beta_Q"'] \arrow[r, mapsto, "\alpha_Q"] \arrow[dl, mapsto, "\tau_{T^*Q}"] \arrow[dr, mapsto, "T\pi_Q"] & (x^i, \dot x^j, \dot p_{x^l}, p_{x^k}) \arrow[d, mapsto, "\pi_{TQ}"] \\
(x^i, p_{x^k}) \arrow[r, mapsto, "\pi_Q"'] & (x^i) & (x^i, \dot x^j) \arrow[l, mapsto, "\tau_Q"]
\end{tikzcd}
$$

Let $L : TQ \to \mathbb{R}$ be the Lagrangian of a mechanical system. The image $dL(TQ)$ of its differential is a Lagrangian submanifold of $\bigl(T^*(TQ), \omega_{TQ}\bigr)$.

Therefore $D = {\alpha_Q}^{-1}\bigl(dL(TQ)\bigr)$ is a Lagrangian submanifold of $\bigl(T(T^*Q), {\alpha_Q}^*(\omega_{TQ})\bigr)$.

Theorem (Tulczyjew). The *Lagrangian submanifold* $D = {\alpha_Q}^{-1}\bigl(dL(TQ)\bigr)$ of $T(T^*Q)$ is the (maybe implicit) *differential equation* which describes the motion of the mechanical system on its *phase space* T^*Q.

In other words, a smooth parametrized curve $c : t \mapsto c(t)$ in Q is stationary for the action integral

$$S(c) = \int_{t_0}^{t_1} L\left(\frac{dc(t)}{dt}\right)\,dt$$

with respect to variations of c with fixed endpoints, if and only if the image $\mathcal{L}_L\left(\frac{dc(t)}{dt}\right)$ of the curve $t \mapsto \frac{dc(t)}{dt}$ by the Legendre map $\mathcal{L}_{:}TQ \to T^*Q$, is such that at each point, the tangent vector

$$\frac{d}{dt}\mathcal{L}_L\left(\frac{dc(t)}{dt}\right)$$

lies in D.

Using the expression of the isomorphism α_Q in local coordinates, the verification of that property is easy (for a coordinate-free proof see the book and the papers by Tulczyjev). The submanifold D is the subset of $T(T^*Q)$

$$D = \left\{ \left(x^i, p_{x^k} = \frac{\partial L(x,v)}{\partial v^k}, \dot{x}^j, \dot{p}_{x^l} = \frac{\partial L(x,v)}{\partial x^l} \right) \right\} .$$

But

$$\frac{dx^i(t)}{dt} = \dot{x}^i , \qquad \frac{dp_{x^k}(t)}{dt} = \dot{p}_{x^k} ,$$

which proves that the expression of D in local coordinates means that we have the Lagrange equation :

$$\frac{d}{dt}\left(\frac{\partial L(x,v)}{\partial v^k}\left(x(t), \frac{dx(t)}{dt} \right) \right) = \frac{\partial L(x,v)}{\partial x^k} \left(x(t), \frac{dx(t)}{dt} \right) .$$

We now assume that the Lagrangian L is hyperregular.The Legendre map $\mathcal{L}_L : TQ \to T^*Q$ is a diffeomorphism and there exists a smooth function (the Hamiltonian) $H : T^*Q \to \mathbb{R}$ such that, in local coordinates,

$$H(x,p) = \left(\sum_{i=1}^{n} v^i \frac{\partial L(x,v)}{\partial v^i} - L(x,v) \right) \circ {\mathcal{L}_L}^{-1}(x,p)\,.$$

Theorem (Tulczyjew). The image $\beta_Q(D)$ of the Lagrangian submanifold D of $T(T^*Q)$ is equal to $-dH(T^*Q)$. More precisely,

$$\beta_Q \circ {\alpha_Q}^{-1} \circ dL = -dH \circ \mathcal{L}_L\,.$$

Using the expressions of α_Q and β_Q in local coordinates given above, it is easy to prove that formula.

W.M. Tulczyjew has considered more general situations :

- situations in which le Lagrangian L is not assumed to be hyperregular,
- or in which L is a *“constrained Lagrangian”*, defined on a submanifold of TQ,
- or in which it is the Hamiltonian H which is *constrained*, defined on a submanifold of T^*Q *(Dirac theory of constraints)*.
- or mechanical systems with external forces (paper [5] by G. Marmo, W.M Tulczyjev and P. Urbański) $\ldots$

Thanks

I address my warmest thanks to the organizers of the Conference “Geometry of Manifolds and Mathematical Physics” for their kind invitation. I am glad to take part in this conference in honour of my colleague and friend Wlodzimierz M. Tulczyjew, whose works were for me a constant source of inspiration. I wish him a happy birthday and many more good years !

And all my thanks to the participants for their interest in my talk !

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Appendix A. The virtual work of accelerations (1).

Following Lagrange, we will denote by δq the virtual infinitesimal displacement, although this notation is misleading : *it is not a differential form*, but rather a *vector field* tangent to the configuration space-time $\widetilde{Q}$ along the the curve $\{c(t); t \in \mathcal{T}\}$.

Moreover, its projection onto $\mathcal{T}$ must vanish : for each $t \in \mathcal{T}$, we must have

$$T_{c(t)}\theta\Bigl(\delta q\bigl(c(t)\bigr)\Bigr) = 0\,.$$

This condition expresses the fact that at each time t, the virtual infinitesimal displacement only affects the configuration of the system, not the time t.

The tangent bundle $T\mathcal{E}$ being trivial, we identify it with $\mathcal{E} \times \overrightarrow{\mathcal{E}}$ and we denote by $\mathrm{pr}_2 : T\mathcal{E} = \mathcal{E} \times \overrightarrow{\mathcal{E}} \to \overrightarrow{\mathcal{E}}$ the second projection. We set

$$\overrightarrow{Z}_\alpha = \mathrm{pr}_2 \circ TM_\alpha : T\widetilde{Q} \to \overrightarrow{\mathcal{E}}\,.$$

Appendix A. The virtual work of accelerations (2).

The virtual infinitesimal work of $\dfrac{d\overrightarrow{p}_\alpha(t)}{dt}$ is

$$\mathcal{W}\left(\frac{d\overrightarrow{p}_\alpha(t)}{dt}\,,\ \delta q\right) = \left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle\,.$$

The pairing $\langle\ ,\ \rangle$ on the left-hand side of this formula stands for the Euclidean scalar product of vectors in $\overrightarrow{\mathcal{E}}$.

The calculation made by Lagrange aims at expressing this infinitesimal virtual work as the pairing of the vector $\delta q\bigl(c(t)\bigr) \in T_{c(t)}\widetilde{Q}$ with a covector, element of $T^*_{c(t)}\widetilde{Q}$.

Lagrange writes

$$\left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle = \frac{d}{dt}\langle \overrightarrow{p}_\alpha(t), \overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\rangle$$

$$- \left\langle \overrightarrow{p}_\alpha(t), \frac{d}{dt}\Bigl(\overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\Bigr)\right\rangle\,.$$

Appendix A. The virtual work of accelerations (3).

Important remark The virtual infinitesimal displacement δq is initially defined as a vector field tangent to $\widetilde{Q}$ along the curve $\{c(t); t \in \mathcal{T}\}$. However, by writing

$$\left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle = \frac{d}{dt}\bigl\langle \overrightarrow{p}_\alpha(t), \overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\bigr\rangle - \left\langle \overrightarrow{p}_\alpha(t), \frac{d}{dt}\Bigl(\overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\Bigr) \right\rangle ,$$

one assumes that δq is a vector field on $T\widetilde{Q}$, projectable on $\widetilde{Q}$ by the map $T\tau_{\widetilde{Q}} : T(T\widetilde{Q}) \to T\widetilde{Q}$, its projection being the vector field δq initially defined on $\widetilde{Q}$ along the curve $\{c(t); t \in \mathcal{T}\}$. At a given time t, each term of the right hand side depends on the value of the derivative $\dfrac{d\bigl(\delta q \circ c(t)\bigr)}{dt}$, but the right hand side as a whole only depends on the value of $\delta q \circ c(t)$.

Appendix A. The virtual work of accelerations (4).

With the local coordinates $(t, q^1, \ldots, q^n, \dot t, \dot q^1, \ldots, \dot q)$ on $T\widetilde{Q}$, we may write

$$\overrightarrow{Z}_\alpha(t, q^1, \ldots, q^n, \dot t, \dot q^1, \ldots, \dot q^n) = \sum_{i=1}^n \dot q^i \, \frac{\partial \overrightarrow{M}_\alpha}{\partial q^i} + \dot t \, \frac{\partial \overrightarrow{M}_\alpha}{\partial t}$$
$$= \sum_{i=1}^n \dot q^i \, \frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot q^i} + \dot t \, \frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot t} \, ,$$

the second equality following from *Euler's identity*, which can be used since $\overrightarrow{Z}_\alpha(t, q^1, \ldots, q^n, \dot t, \dot q^1, \ldots, \dot q^n)$ is a linear function of $(\dot t, \dot q^1, \ldots, \dot q^n)$.Thefrefore,

$$\frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot q^i} = \frac{\partial \overrightarrow{M}_\alpha}{\partial q^i} \, , \quad \frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot t} = \frac{\partial \overrightarrow{M}_\alpha}{\partial t} \, .$$

Appendix A. The virtual work of accelerations (5).

We may therefore write

$$\left\langle \overrightarrow{p}_\alpha(t), \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle = m \sum_{i=1}^{n} \left\langle \overrightarrow{Z}_\alpha \circ \frac{dc(t)}{dt}, \frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \bigl(\delta q^i \circ c(t)\bigr) \right\rangle$$

with, in local coordinates,

$$\frac{dc(t)}{dt} = \left(t, q^1(t), \dots, q^n(t), 1, \frac{dq^1(t)}{dt} \dots, \frac{dq^n(t)}{dt} \right) .$$

Let $T_\alpha : T\widetilde{Q} \to \mathbb{R}$ be the function

$$T_\alpha = \frac{m_\alpha}{2} \langle \overrightarrow{Z}_\alpha, \overrightarrow{Z}_\alpha \rangle . \quad \text{We have :}$$

$$\left\langle \overrightarrow{p}_\alpha(t), \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle = \sum_{i=1}^{n} \left(\frac{\partial T_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \right) \bigl(\delta q^i \circ c(t)\bigr) .$$

Appendix A. The virtual work of accelerations (6).

Taking the derivative with respect to t, we get

$$\frac{d}{dt}\langle \overrightarrow{p}_\alpha(t), \overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\rangle = \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial T_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt}\right)\left(\delta q^i \circ c(t)\right) + \sum_{i=1}^{n}\left(\frac{\partial T_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt}\right)\frac{d}{dt}\left(\delta q^i \circ c(t)\right).$$

Similarly, we may write

$$\left\langle \overrightarrow{p}_\alpha(t), \frac{d}{dt}\left(\overrightarrow{Z}_\alpha \circ \delta q \circ c(t)\right)\right\rangle = \left\langle \overrightarrow{p}_\alpha(t), \sum_{i=1}^{n} \frac{d}{dt}\left(\frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\left(c(t)\right)\delta q^i\left(c(t)\right)\right)\right\rangle$$

$$= \left\langle \overrightarrow{p}_\alpha(t), \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\left(c(t)\right)\right)\delta q^i\left(c(t)\right) + \frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\left(c(t)\right)\frac{d}{dt}\left(\delta q^i\left(c(t)\right)\right)\right]\right\rangle$$

Appendix A. The virtual work of accelerations (7).

But we have

$$\frac{d}{dt}\left(\frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\big(c(t)\big)\right) = \frac{\partial}{\partial q^i}\left(\frac{d\overrightarrow{M}_\alpha\big(c(t)\big)}{dt}\right) = \frac{\partial \overrightarrow{Z}_\alpha}{\partial q^i}\left(\frac{dc(t)}{dt}\right).$$

Therefore

$$\begin{aligned}\left\langle \overrightarrow{p}_\alpha(t), \sum_{i=1}^{n}\frac{d}{dt}\left(\frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\big(c(t)\big)\right)\delta q^i\big(c(t)\big)\right\rangle \\ = \left\langle m\overrightarrow{Z}_\alpha \circ \frac{dc(t)}{dt}, \sum_{i=1}^{n}\frac{\partial \overrightarrow{Z}_\alpha}{\partial q^i}\circ\frac{dc(t)}{dt}\delta q^i \circ c(t)\right\rangle \\ = \sum_{i=1}^{n}\left(\frac{\partial T_\alpha}{\partial q^i}\circ\frac{dc(t)}{dt}\right)\delta q^i \circ c(t).\end{aligned}$$

Appendix A. The virtual work of accelerations (8).

The last term can be written

$$\left\langle \overrightarrow{p}_\alpha(t), \sum_{i=1}^{n} \frac{\partial \overrightarrow{M}_\alpha}{\partial q^i}\big(c(t)\big) \frac{d}{dt}\Big(\delta q^i\big(c(t)\big)\Big) \right\rangle$$

$$= \sum_{i=1}^{n} \left[\left\langle m\overrightarrow{Z}_\alpha \circ \frac{dc(t)}{dt}, \frac{\partial \overrightarrow{Z}_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \right\rangle \frac{d}{dt}\Big(\delta q^i\big(c(t)\big)\Big) \right]$$

$$= \sum_{i=1}^{n} \left(\frac{\partial T_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \right) \frac{d}{dt}\Big(\delta q^i\big(c(t)\big)\Big) .$$

When we gather all the terms calculated, we see that the terms which contain $\dfrac{d}{dt}\Big(\delta q^i\big(c(t)\big)\Big)$ cancel. The virtual work of $\dfrac{d\overrightarrow{p}_\alpha(t)}{dt}$ for the infinitesimal virtual displacement δq is :

Appendix A. The virtual work of accelerations (9).

$$\mathcal{W}\left(\frac{d\overrightarrow{p}_\alpha(t)}{dt}, \ \delta q\right) = \left\langle \frac{d\overrightarrow{p}_\alpha(t)}{dt}, \overrightarrow{Z}_\alpha \circ \delta q \circ c(t) \right\rangle$$
$$= \sum_{i=1}^{n} \left[\left(\frac{d}{dt}\left(\frac{\partial T_\alpha}{\partial \dot{q}^i} \circ \frac{dc(t)}{dt} \right) - \frac{\partial T_\alpha}{\partial q^i} \circ \frac{dc(t)}{dt} \right) \left(\delta q^i \circ c(t)\right) \right] .$$

This virtual work is expressed as the pairing of the vector $\delta q \circ c(t) \in T_{c(t)}\widetilde{Q}$ with a covector, element of $T^*_{c(t)}\widetilde{Q}$. More exactly, since $\delta q \circ c(t) \in \ker T_{c(t)}\theta$, that covector is determined only up to addition of any covector which vanishes on $\ker T_{c(t)}\theta$; in other words it is an element of the quotient space $T^*_{c(t)}\widetilde{Q}/\bigl(\ker T_{c(t)}\theta\bigr)$.

Appendix B. The homogeneous Lagrangian

We recall that the Lagrangian L is defined on the codimension 1 submanifold $T^1\widetilde{Q}$ of $T\widetilde{Q}$ of vectors whose projection on the time axis $\mathcal{T}$ is equal to 1. The action integral

$$S(c) = \int_{t_0}^{t_1} L \circ \frac{dc(t)}{dt}\, dt$$

is defined for smooth sections c of $\theta : \widetilde{Q} \to \mathcal{T}$, *i.e.* for curves in $\widetilde{Q}$ parametrized by the time. It is easy to extend the definition of the Lagarangian to an open dense subset of $T\widetilde{Q}$ in such a way that the action integral still has a meaning for geometric smooth curves in $\widetilde{Q}$, independent of their parametrization. With $(t, q^1 \ldots, q^n, \dot{t}, \dot{q}^1, \ldots, \dot{q}^n)$ as local coordinates on $T\widetilde{Q}$, let

$$\widehat{L}(t, q^1 \ldots, q^n, \dot{t}, \dot{q}^1, \ldots, \dot{q}^n) = \dot{t} L\left(t, q^1 \ldots, q^n, 1, \frac{\dot{q}^1}{\dot{t}}, \ldots, \frac{\dot{q}^n}{\dot{t}}\right) .$$

Appendix B. The homogeneous Lagrangian (2)

The function $\widehat{L}$, defined on the open dense subset of $T\widetilde{Q}$ on which the local coordinate $\dot{t}$ is not zero, is homogenous of degree 1 on the fibres. Let $\widehat{c} : [s_0, s_1] \to \widetilde{Q}$ be a smooth parametrized curve such that $s \mapsto \theta \circ \widehat{c}(s)$ is a diffeomorphism of the open interval $]s_0, s_1[$ onto an open interval of the time axis $\mathcal{T}$. In other words, we assume that for any $s \in]s_0, s_1[$,

$$\frac{d}{ds}\bigl(\theta \circ \widehat{c}(s)\bigr) \neq 0\,.$$

Such a curve will be said to be *admissible*.

We define a *modified action integral*

$$\widehat{S}(\widehat{c}) \int_{s_0}^{s_1} \widehat{L}\left(\frac{d\widehat{c}(s)}{ds}\right)\, ds\,.$$

Appendix B. The homogeneous Lagrangian (3)

Since $\widehat{L}$ is homogeneous of degree 1, $\widehat{S}(\widehat{c})$ only depends on the geometric curve $\widehat{c}\bigl([s_0, s_1]\bigr)$, not on its parametrization. When $[s_0, s_1]$ is an interval of $\mathcal{T}$ and $\widehat{c}$ a section of θ, $\widehat{S}(\widehat{c}) = S(\widehat{c})$.

The *vertical differential* $d_V\widehat{L}$ of the homogeneous Lagrangian $\widehat{L}$ is a 1-form defined on the open dense subset of $T\widehat{Q}$ on which $\widehat{L}$ is defined. It is called the *Hilbert's 1-form* in the book [4] by *P. Malliavin*. In local coordinates

$$\varpi = d_V\widehat{L} = \frac{\partial \widehat{L}}{\partial t}\,dt + \sum_{i=1}^{n} \frac{\partial \widehat{L}}{\partial \dot{q}^i}\,dq^i\,.$$

It is such that for any admissible parametrized curve $\widehat{c} : [s_0, s_1] \to \widetilde{Q}$,

$$\widehat{S}(\widehat{c}) = \int \left(\frac{d\widehat{c}(s)}{ds}\right)^{*} \varpi\,.$$

Appendix B. The homogeneous Lagrangian (4)

The 1-form $\sigma = i^*_{T^1\widetilde{Q}}\varpi$ induced by ϖ on the codimension 1 submanifold $T^1\widetilde{Q}$ is expressed, with the local coordinates $(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$, as

$$\sigma = i^*_{T^1\widetilde{Q}}\varpi = \sum_{i=1}^{n} \frac{\partial L(t,q,\dot{q})}{d\dot{q}^i}\, dq^i - E(t,q,\dot{q})\, dt\,,$$

where $E(t,q,\dot{q})$ is the *energy function*, given by

$$E(t,q,\dot{q}) = \sum_{i=1}^{n} \dot{q}^i \frac{\partial L(t,q,\dot{q})}{\partial \dot{q}^i} - L(t,q,\dot{q})\,.$$

For any smooth section $c : [t_0, t_1] \to \widetilde{Q}$ of θ

$$S(c) = \int \left(\frac{dc(t)}{dt}\right)^* \sigma\,.$$

Appendix B. The homogeneous Lagrangian (5)

By using the fact that an admissible parametrized curve $\widehat{c} : [s_0, s_1] \to \widetilde{Q}$ satisfies the *principle of virtual work* if and only if the modified action $\widehat{S}(\widehat{c})$ is stationary for all infinitesimal variations of $\widehat{c}$ with fixed endpoints, we see that such a curve satisfies that principle if and only if, for each $s \in]s_0, s_1[$,

$$i\left(\frac{d^2\widehat{c}(s)}{ds^2}\right) d\varpi = 0\,.$$

Similarly, a smooth section $c : [t_0, t_1] \to \widetilde{Q}$ satisfies the *principle of virtual work* if and only if, for each $t \in]t_0, t_1[$,

$$i\left(\frac{d^2c(t)}{dt^2}\right) d\sigma = 0\,.$$

This equation is the *intrisic form of the Lagrange equations*.

Appendix B. The homogeneous Lagrangian (6)

The *Legendre map* can be defined either with the *orignial Lagarangian* L, or with the *homogeneous Lagrangian* $\widehat{L}$. We will denote these two Legendre maps $\mathcal{L}_L$ and $\mathcal{L}_{\widehat{L}}$, respectively.

Let us first consider $\mathcal{L}_{\widehat{L}} : T\widetilde{Q} \to T^*\widetilde{Q}$. In local coordinates $(t, q^i, \dot{t}, \dot{q}^i)$ on $T\widetilde{Q}$ and (t, q^i, p_t, p_i) on $T^*\widetilde{Q}$, $1 \leq i \leq n$, it is the map

$$\mathcal{L}_{\widehat{L}} : (t, q^i, \dot{t}, \dot{q}^i) \mapsto \left(t, q^i, p_t = \frac{\partial \widehat{L}(t, q^i, \dot{t}, \dot{q}^i)}{\partial \dot{t}}, p_i = \frac{\partial \widehat{L}(t, q^i, \dot{t}, \dot{q}^i)}{\partial \dot{q}^i} \right) .$$

Using the definition of $\widehat{L}$ in terms of L, we have

$$\frac{\partial \widehat{L}(t, q^i, \dot{t}, \dot{q}^i)}{\partial \dot{t}} = -E\left(t, q^i, \frac{\dot{q}^i}{\dot{t}}\right) , \qquad \frac{\partial \widehat{L}(t, q^i, \dot{t}, \dot{q}^i)}{\partial \dot{q}^i} = \frac{\partial L}{\partial \dot{q}^i}\left(t, q^i, \frac{\dot{q}^i}{\dot{t}}\right) ,$$

where E is the *energy function*

Appendix B. The homogeneous Lagrangian (7)

Therefore, expressed in terms of L and E,

$$\mathcal{L}_{\widehat{L}} : (t, q^i, \dot{t}, \dot{q}^i) \mapsto \left(t, q^i, p_t = -E\left(t, q^i, \frac{\dot{q}^i}{\dot{t}}\right), p_i = \frac{\partial L}{\partial \dot{q}^i}\left(t, q^i, \frac{\dot{q}^i}{\dot{t}}\right)\right) .$$

The Legendre map $\mathcal{L}_{\widehat{L}}$ cannot be a local diffeomorphism : *its rank is at most equal to* $2n+1$, since its values only depend on the ratios $\dfrac{\dot{q}^i}{\dot{t}}$.

The Lagrangian L is said to be *regular* if the Legendre map $\mathcal{L}_{\widehat{L}}$ is everywhere of rank $2n+1$; its restriction to the submanifold $T^1\widetilde{Q}$ of $T\widetilde{Q}$ is then a local diffeomorphism of $T^1\widetilde{Q}$ on its image.

The Lagrangian L is said to be *hyperregular* if $\mathcal{L}_{\widehat{L}}$, restricted to $T^1\widetilde{Q}$, is a global diffeomorphism of $T^1\widetilde{Q}$ onto its image.

Appendix B. The homogeneous Lagrangian (8)

Hilbert's 1-form ϖ was defined above as the *vertical differential* $d_V\widehat{L}$ of the homogeneous Lagrangian. One may check that it can be defined also as the *pull-back* of the Liouville 1-form $\eta_{\widetilde{Q}}$ of $T^*\widetilde{Q}$ by the Legendre map $\mathcal{L}_{\widehat{L}}$:

$$\varpi = d_V\widehat{L} = \mathcal{L}^*_{\widehat{L}}(\eta_{\widetilde{Q}})\,.$$

We recall that σ is the 1-form induced by ϖ on the submanifold $T^1\widetilde{Q}$. When L is regular, $\mathcal{L}_{\widehat{L}}$ restricted to $T^1\widetilde{Q}$ is a local diffeomorphism of $T^1\widetilde{Q}$ on its image, which therefore is an immersed submanifold (maybe with self intersections) of $T^*\widetilde{Q}$, *coisotropic* since its codimension is 1. Therefore $d\sigma$ is of rank $2n$, and there exists on $T^1\widetilde{Q}$ a unique vector field $\mathcal{X}_L$ contained in $\ker d\sigma$ whose projection on $\mathcal{T}$ is equal to 1. Integral curves of this vector field are *motions of the mechanical system*.

Appendix B. The homogeneous Lagrangian (9)

Still when L is regular, the *manifold of motions* of the mechanical system is the quotient of the presymplectic manifold $(T^1\widetilde{Q}, d\sigma)$ by its characteristic foliation determined by $\ker d\sigma$.
J. M. Souriau [7] has shown that it has indeed the structure of a *smooth symplectic manifold* (maybe non-Hausdorff).
The Legendre map $\mathcal{L}_L$ defined with the original Lagrangian L, expressed in local coordinates $(t, q^i, \dot{q}^i)$ on $T^1\widetilde{Q}$ (submanifold of $T\widetilde{Q}$ on which $\dot{t} = 1$) is

$$\mathcal{L}_L : (t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_i = \frac{\partial L(t, q^i, \dot{q}^i)}{\partial \dot{q}^i} \right) .$$

It is defined on $T^1\widetilde{Q}$, and takes its values in the quotient bundle $T^*\widetilde{Q}/(\ker T\theta)^0$. Its use is interesting when a trivialization of the time-configuration manifold $\widetilde{Q}$ into a product $\mathcal{T} \times Q$ of the time axis and an n-dimensional configuration manifold Q is chosen.

Appendix B. The homogeneous Lagrangian (10)

We now assume that $\widetilde{Q} = \mathcal{T} \times Q$, where $\mathcal{T}$ is the time axis and Q a *configuration manifold*. The map $\theta : \widetilde{Q} \to \mathcal{T}$ is the first projection. The codimension 1 submanifold $T^1\widetilde{Q}$ can be identified with $\mathcal{T} \times TQ$, and the quotient manifold $T^*\widetilde{Q}/(\ker T\theta)^0$ with $\mathcal{T} \times T^*Q$. The *Legendre map* determined by the Lagrangian L can therefore be considered as a map $\mathcal{L}_L : \mathcal{T} \times TQ \to \mathcal{T} \times T^*Q$,

$$\mathcal{L}_L : (t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_i = \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \right) , \quad 1 \leq i \leq n_, .$$

The cotangent bundle $T^*\widetilde{Q}$ can be identified with $T^*\mathcal{T} \times T^*Q$, and the *Legendre map* determined by the homogeneous Lagrangian $\widehat{L}$, restricted to $T^1\widetilde{Q} = \mathcal{T} \times TQ$, is

$$\mathcal{L}_{\widehat{L}} \Big|_{T^1\widetilde{Q}} : (t, q^i, \dot{q}^i) \mapsto \left(t, q^i, p_t = -E(t, q^i, \dot{q}^i), p_i = \frac{\partial L(t, q, \dot{q})}{\partial \dot{q}^i} \right) .$$

Appendix B. The homogeneous Lagrangian (11)

Therefore

$$\mathcal{L}_{\widehat{L}} \Big|_{T^1\widetilde{Q}} = \mathcal{L}_L - E\, dt\, .$$

Regularity and hyperregularity of the Lagrangian L, defined above in terms of properties of $\mathcal{L}_{\widehat{L}}$, may be seen also by properties of $\mathcal{L}_L$: the Lagrangian L is *regular* if the Legendre map $\mathcal{L}_L$ is a local diffeomorphism and *hyperregular* if $\mathcal{L}_L$ is a global diffeomorphism.

Appendix B. The homogeneous Lagrangian (12)

We still assume that $\widetilde{Q} = \mathcal{T} \times Q$ and, in addition, that the Lagrangian L is *hyperregular*. We have seen that the *motions* of the mechanical system are integral curves of a vector field $\mathcal{X}_L$, defined on $T^1\widetilde{Q} = \mathcal{T} \times TQ$, such that

$$i(\mathcal{X}_L)d\sigma = 0\,, \quad T\theta(\mathcal{X}_L) = 1\,,$$

(the meaning of 1 in the right hand side is the constant vector field of unit length on $\mathcal{T}$).

The image $W = \mathcal{L}_{\widehat{L}}(T\widetilde{Q})$ of the Legendre map $\mathcal{L}_{\widehat{L}}$ is a codimension-1 submanifold of $T^*\widetilde{Q}$, on which we can define the vector field

$$\mathcal{Y}_L = (\mathcal{L}_{\widehat{L}})_*(\mathcal{X}_L)\,,$$

direct image of the vector field $\mathcal{X}_L$ by the diffeomorphism $\mathcal{L}_{\widehat{L}}\,\big|_{T^1\widetilde{Q}} \colon T^1\widetilde{Q} \to W$.

Appendix B. The homogeneous Lagrangian (13)

The vector field $\mathcal{Y}_L$ is determined by the conditions

$$i(\mathcal{Y}_L)d(i_W^*\eta_{\widetilde{Q}}) = 0\,, \quad T\pi_{\mathcal{T}}(\mathcal{Y}_L) = 1\,,$$

where $i_W^*\eta_{\widetilde{Q}}$ is the form induced on W by the Liouville 1-form of $T^*\widetilde{Q}$, and $\pi_{\mathcal{T}} : W \to \mathcal{T}$ the natural projection on the time axis $\mathcal{T}$. The *Hamiltonian* is the function

$$H = E \circ \mathcal{L}_L^{-1} : \mathcal{T} \times T^*Q \to \mathbb{R}\,.$$

The map

$$(t, q^i, p_i) \mapsto (t, q^i, p_t = H(t, q^i, p_i))\,, \quad 1 \leq i \leq n\,,$$

allows us to identify $\mathcal{T} \times T^*Q$ with the submanifold W of $T^*\widetilde{Q}$.

Appendix B. The homogeneous Lagrangian (14)

Using this identification of $\mathcal{T} \times T^*Q$ with W, the form induced on W by the Liouville 1-form of $T^*\widetilde{Q}$ becomes the form on $\mathcal{T} \times T^*Q$

$$\eta_Q - H\,dt\,,$$

where η_Q is the Liouville 1-form on T^*Q

The vector field $\mathcal{Y}_L$, now considered as defined on $\mathcal{T} \times T^*Q$, is therefore determined by

$$i(\mathcal{Y}_L)(d\eta_Q - dH \wedge dt) = 0\,, \quad T\pi_{\mathcal{T}}(\mathcal{Y}_L) = 1\,.$$

The second equality above allows us to write

$$\mathcal{Y}_L = X_H + \frac{\partial}{\partial t}\,,$$

where X_H is a *time-dependent vector field* on T^*Q.

Appendix B. The homogeneous Lagrangian (15)

The first equality determining $\mathcal{Y}_L$ leads to

$$i(X_H)d\eta_Q = -(dH - \frac{\partial H}{\partial t}\,dt)\,, \quad i(X_H)dH = 0\,.$$

The first equation shows that for each fixed time t, the value X_{H_t} of the time-dependent vector field X_H is the Hamiltonian vector field on T^*Q whose Hamiltonian is $H_t : T^*Q \to \mathbb{R}$. The second equation is automatically satisfied when the first equation is satisfied.

This is the *Hamiltonian formalism*, equivalent to the Lagrangian formalism when the Lagrangian L is hyperregular.