Mechanics in Space-Time, Connections and the Principle of Inertia

Charles-Michel Marle

cmm1934@orange.fr

Université Pierre et Marie Curie

Paris, France

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- a material body is *at rest* if its position in Space does not depend on Time ;
- Time can be mathematically modelled by an real, affine, one-dimensional space $\mathcal{T}$;
- Space can be mathematically modelled by an affine, real, Euclidean (once a unit of length has been chosen), three-dimensional space $\mathcal{E}$.

I. Introduction (2).

With these assumptions, the fundamental law which describes the motion of a material point of mass m submitted to a force $\overrightarrow{F}$, can be written

$$\overrightarrow{F(t)} = m\frac{\overrightarrow{d^2x(t)}}{dt^2}\,.$$

In this equation, t is an element of Time $\mathcal{T}$ (identified with the real line $\mathbb{R}$ by the choice of an orgin and a unit of Time) and $x(t)$ is an element of Space $\mathcal{E}$, the *position in Space* at Time t of the material point under consideration.

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$\overrightarrow{F(t)}$ is the force which, at Time t, acts on that material point.

I. Introduction (3).

The first and second derivatives

$$\overrightarrow{v(t)} = \frac{\overrightarrow{dx(t)}}{dt} \quad \text{and} \quad \overrightarrow{a(t)} = \frac{\overrightarrow{d^2x(t)}}{dt^2}$$

of the position $x(t)$ with respect to t are, respectively, the *velocity* and the *acceleration* of the material point. They live in *different spaces* : the tangent space $T_{x(t)}\mathcal{E}$ at $x(t)$ to $\mathcal{E}$, and the tangent space $T_{\overrightarrow{v(t)}}(T\mathcal{E})$ at $\overrightarrow{v(t)}$ to the tangent bundle $T\mathcal{E}$. It is the *triviality* of the tangent bundle $T\mathcal{E}$, due to the affine structure of $\mathcal{E}$, which allows to consider them both as elements of the Euclidean vector space $\overrightarrow{\mathcal{E}}$ associated to the affine Euclidean space $\mathcal{E}$.

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The *force* $\overrightarrow{F(t)}$ which, at each time t, acts on the material point is, too, an element of $\overrightarrow{\mathcal{E}}$.

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However, the position in Space of a material body can be appreciated only *with respect to other bodies* (material or conceptual). Newton considered that the centre of the Sun (or, maybe, the centre of mass of the Solar system) and the straight lines which join that point to distant stars are at rest.

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Moreover, Newton observed that since the velocity $\overrightarrow{\dfrac{dx(t)}{dt}}$ does not appear in the equation

$$\overrightarrow{F(t)} = m\overrightarrow{\frac{d^2x(t)}{dt^2}}\,,$$

that equation remains unchanged if $x(t)$, instead of being the absolute position in Space of the moving material point at time t, is its *relative position* at that time with respect to a reference frame whose motion in Space is a translation at a constant velocity.

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But do the the notions of *absolute rest of a material body* and of *absolute motion of a reference frame* in Space have really a meaning ?

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If, following Leibniz, we consider that the notion of absolute rest in Space does not mean anything, we must reconsider the mathematical modelizations of Space and Time, and formulate the definition of inertial frames without any reference to their absolute motion in Space.

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In what follows we will look at several possible mathematical modelizations of Space and Time and at the corresponding mathematical formulations of the laws of Dynamics.

- First we will look at models in which the assumptions made about Time and Space are *global*,
- and then at models in which these assumptions are only *local*. Such models are in better agreement with the wiews presented by *Bernhard Riemann* in his famous inaugural lecture *Sur les hypothèses qui servent de fondement à la géométrie* [6].

II. Classical Mechanics. 1. Leibniz Space-Time

In classical (non-relativistic) Mechanics, Time has an absolute meaning : its flow is everywhere the same. In other words, all events which happen in the Universe can be chronologically ordered. Time is mathematically modelled by a real affine one-dimensional space $\mathcal{T}$.

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If we consider that the notions of *absolute rest* and of *absolute motion* have no meaning, Space cannot be mathematically modelled by an unique Euclidean space. We must consider that for each time $t \in \mathcal{T}$, there exists a *Space at time* t, $\mathcal{E}_t$. Observe that if t_1 and t_2 are two distinct elements of Time $\mathcal{T}$, the spaces $\mathcal{E}_{t_1}$ and $\mathcal{E}_{t_2}$ are *disjoint*.

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The concept of *fibered space* is the tool able to offer, in Classical Mechanics, a mathematical model of Space and Time together. Unfortunately that concept did not exist when Newton and Leibniz had a controversy about the existence of an Absolute Space.

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In Classical Mechanics, Space-Time is mathematically modelled by a four-dimensional manifold wich will be denoted by $\mathcal{L}$ (in honour of Leibniz), fibered over the Time $\mathcal{T}$. The fibration

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For each time $t \in \mathcal{T}$, the fibre $\mathcal{E}_t = \theta^{-1}(t)$ is the *Space at time t*. In Classical Mechanics, one assumes that each $\mathcal{E}_t$ is a real, affine, three-dimensional Euclidean space (once a unit of length has been chosen).

II. Classical Mechanics. 2. Reference frames

The *standard fibre* of the fibration $\theta : \mathcal{L} \to \mathcal{T}$ of the Leibniz Space-Time is a real affine Euclidean (once a unit of length is chosen) three-dimensional space $\mathcal{E}$. It is a mathematical abstraction, not a real physical object. The way in which for each time $t \in \mathcal{T}$, the Space at time t, $\mathcal{E}_t$, is identified with the standard fibre $\mathcal{E}$, depends on the choice of a *reference frame*.

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A *reference frame* in Classical Mechanics is a trivialization

$$\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E} \quad \text{of the fibration} \quad \theta : \mathcal{L} \to \mathcal{T}\,,$$

that means a diffeomorphism such that $p_1 \circ \Phi = \theta$ and that for each $t \in \mathcal{T}$, $p_2 \circ \Phi|_{\mathcal{E}_t} : \mathcal{E}_t \to \mathcal{E}$ is an isometry of affine Euclidean spaces.

We have denoted by p_1 and p_2 the projections of $\mathcal{T} \times \mathcal{E}$ on $\mathcal{T}$ and $\mathcal{E}$, respectively.

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For Newton, who believed in the existence of an Absolute Space $\mathcal{E}$, the motion of a material point was mathematically represented by a continuous map $t \mapsto x(t)$, defined on Time $\mathcal{T}$ (or on an interval of $\mathcal{T}$), with values in $\mathcal{E}$. The *Principle of Inertia* was the following statement :

> Any *free material point* moves on a *straight line* in Space $\mathcal{E}$ at a *constant velocity.*

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> Any *free material point* moves on a *straight line* in Space $\mathcal{E}$ at a *constant velocity.*

When one no more assumes the existence of an *absolute Space*, $\mathcal{E}$ has no more a physical existence : it is only a mathematical concept, the *standard fibre* of the fibered Leibniz Space-Time $\theta : \mathcal{L} \to \mathcal{T}$.

The motion of a material point is now mathematically represented by a continuous section $t \mapsto z(t)$ of the fibration $\theta : \mathcal{L} \to \mathcal{T}$, defined on Time $\mathcal{T}$ (or on an interval of $\mathcal{T}$), with values in Space-Time $\mathcal{L}$. The *relative motion* of that material point with respect to a reference frame $\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E}$ is the map, defined on $\mathcal{T}$, with values in $\mathcal{E}$, $t \mapsto p_2 \circ \Phi \circ z(t)$.

II. Classical Mechanics. 3. The Principle of Inertia (2)

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The *Principle of Inertia* should now be stated as follows :

> There exists on Space-Time at least one *privileged reference frame* $\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E}$, called *inertial*, or *Galilean*, such that the relative motion motion of any *free material point* with respect to that frame occurs on a straight line in the standard fibre $\mathcal{E}$ at a constant velocity.

The existence of *one* inertial frame implies the existence of an *infinite number* of such frames, whose relative motions are *translations at constant velocities*.

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> There exists on Space-Time $\mathcal{L}$ an *affine structure* such that the map $\theta : \mathcal{L} \to \mathcal{T}$ is an *affine map* and that the motion $t \mapsto z(t)$ of any free material point is an *affine map* (defined on $\mathcal{T}$, or on an interval of $\mathcal{T}$, with values in $\mathcal{L}$).

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Let $t \mapsto z(t)$ be the motion of a material point. The *world line* of this material point is the subset $\{z(t) | t \in \mathcal{T}\}$ of Leibniz Space-Time $\mathcal{L}$, and the above statement can be formulated as

> There exists on Space-Time $\mathcal{L}$ an *affine structure* such that the map $\theta : \mathcal{L} \to \mathcal{T}$ is an *affine map* and that the world line of any free material point is a (segment of a) *straight line* in $\mathcal{L}$.

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A reference frame $\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E}$ is *inertial* if and only if it is an *affine spaces isomorphism*.

Let $t \mapsto z(t)$ be the motion of a (maybe non-free) material point : it is a (local) section of the fibration $\theta : \mathcal{L} \to \mathcal{T}$, *i.e.* a map $\varphi : I \to \mathcal{L}$, defined on an open interval I of $\mathcal{T}$, such that $\theta \circ \varphi = \mathrm{id}_I$. If this map is smooth at $t \in I$, once a unit of time has been chosen, the derivative

$$\overrightarrow{w(t)} = \frac{\overrightarrow{dz(t)}}{dt}$$

can be defined. It is the vector, tangent to the world line at $z(t)$, whose projection by $T\theta : T\mathcal{L} \to T\mathcal{T}$ is the unit vector of $\overrightarrow{\mathcal{T}}$. We will call it the *world velocity* of the material point at $z(t)$.

The affine structure of $\mathcal{L}$ determines a natural trivialization of its tangent bundle $T\mathcal{L}$, identified with $\mathcal{L} \times \overrightarrow{\mathcal{L}}$. If the map φ is C^2 at $t \in I$ we can define the *acceleration vector*

$$\overrightarrow{a(t)} = \frac{\overrightarrow{d^2 z(t)}}{dt^2}$$

and consider it as an element of the tangent space $T_{z(t)}\mathcal{L}$, canonically isomorphic to the Euclidean vector space $\overrightarrow{\mathcal{L}}$ asssociated to the affine space $\mathcal{L}$.

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and consider it as an element of the tangent space $T_{z(t)}\mathcal{L}$, canonically isomorphic to the Euclidean vector space $\overrightarrow{\mathcal{L}}$ asssociated to the affine space $\mathcal{L}$.
More precisely, since $T\theta$ maps the world velocity onto the unit vector of $\overrightarrow{\mathcal{T}}$, the acceleration vector $\overrightarrow{a(t)}$ lives in the tangent space $T_{z(t)}\mathcal{E}_t$, subset of the tangent space $T_{z(t)}\mathcal{L}$, canonically isomorphic to the Euclidean vector space $\overrightarrow{\mathcal{E}}$ asssociated to the affine space (standard fibre) $\mathcal{E}$.

The *equation of motion* of the material point can therefore be written as

$$m\frac{\overrightarrow{d^2z(t)}}{dt^2} = \overrightarrow{F}\left(z(t), \frac{\overrightarrow{dz(t)}}{dt}\right),$$

where m is the *mass* of the material point and $\overrightarrow{F}$ the *force* which acts on it, which may depend on its *location in Space-Time* $z(t)$ and on its *world velocity* $\dfrac{\overrightarrow{dz(t)}}{dt}$. At each time t, the force $\overrightarrow{F}$ is an element of $T_{z(t)}\mathcal{E}_t$, the tangent space at $z(t)$ to the fibre $\mathcal{E}_t$.

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Observe that this equation *does not use any reference frame*.

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Observe that this equation *does not use any reference frame*.

The equations of motion of mechanical systems more complicated than material points are usually derived from a *Lagrangian* or from a *Hamiltonian*, which depend on the *choice of a reference frame*.

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A *Galilean transformation* of $\mathcal{T} \times \mathcal{E}$ is an *affine transformation* of that space, $g : (t, x) \mapsto (t', x')$, t and $t' \in \mathcal{T}$, x and $x' \in \mathcal{E}$, such that

- t' does not depend on x, and $t \mapsto t'$ is a *translation* of $\mathcal{T}$,
- for each fixed $t \in \mathcal{T}$, the map $x \mapsto x'$ is an *orientation preserving isometry* of $\mathcal{E}$.

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- for each fixed $t\in\mathcal{T}$, the map $x\mapsto x'$ is an *orientation preserving isometry* of $\mathcal{E}$.

The set of all Galilean transformations of $\mathcal{T}\times\mathcal{E}$ is the *Galilean group* of that space. It is a connected real 10-dimensional Lie group, denoted by $\mathbf{Galileo}(\mathcal{T}\times\mathcal{E})$. Let $\Phi:\mathcal{L}\to\mathcal{T}\times\mathcal{L}$ be an inertial frame. The map

$$(g,z)\mapsto\Phi^{-1}\circ g\circ\Phi(z)\,,\quad z\in\mathcal{L}\,,\quad g\in\mathbf{Galileo}(\mathcal{T}\times\mathcal{E})\,,$$

is an *action* of the Galilean group on Leibniz Space-Time $\mathcal{L}$.

III. Other possible assumptions. 1. A non-Euclidean Space

Global assumptions about *Time* and *Space*, slightly different from those used in *Classical Mechanics*, seem possible.

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For example, we may still assume that *Time* is modelled by a *real, affine one-dimensional space* $\mathcal{T}$, and *Space-Time* by a *real four-dimensional manifold* $\mathcal{U}$, fibered over Time $\mathcal{T}$ by a *date map* $\theta : \mathcal{U} \to \mathcal{T}$. But instead of assuming that the fibres $\theta^{-1}(t)$, $t \in \mathcal{T}$, are real, affine Euclidean three-dimensional spaces, we can assume that these fibres are *real Riemannian three-dimensional homogeneous spaces* : positively curved *spheres* S^3, or negatively curved sheets of *hyperboloids* H^3.

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Let us look at what can be said about the *Principle of Inertia* and about *inertial frames* under such assumptions.

III. Other possible assumptions. 1. A non-Euclidean Space (2)

We now assume that Space-Time is a smooth 4-dimensional manifold $\mathcal{U}$ fibered over Time $\mathcal{T}$ by a *date map* $\theta : \mathcal{U} \to \mathcal{T}$, whose fibres $\theta^{-1}(t)$ are 3-dimensional Riemannian homogeneous spaces of a Lie group G. We assume that G is the group of orientation-preserving isometries of the *standard fibre* $\mathcal{M}$ of the fibered space $\mathcal{U}$.

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A *reference frame* is a trivialization $\Phi : \mathcal{U} \to \mathcal{T} \times \mathcal{M}$ of the fibered space $\mathcal{U}$. The *motion of a material point* is a smooth section $\varphi : I \to \mathcal{U}$ of the fibration θ, defined on an open interval of time $I \subset \mathcal{T}$, with values in $\mathcal{U}$. The *relative motion* of that material point with respect to the reference frame Φ is the map $p_2 \circ \Phi \circ \varphi : I \to \mathcal{M}$, where $p_2 : \mathcal{T} \times \mathcal{M} \to \mathcal{M}$ is the second projection.

III. Other possible assumptions. 1. A non-Euclidean Space (2)

We now assume that Space-Time is a smooth 4-dimensional manifold $\mathcal{U}$ fibered over Time $\mathcal{T}$ by a *date map* $\theta : \mathcal{U} \to \mathcal{T}$, whose fibres $\theta^{-1}(t)$ are 3-dimensional Riemannian homogeneous spaces of a Lie group G. We assume that G is the group of orientation-preserving isometries of the *standard fibre* $\mathcal{M}$ of the fibered space $\mathcal{U}$.

A *reference frame* is a trivialization $\Phi : \mathcal{U} \to \mathcal{T} \times \mathcal{M}$ of the fibered space $\mathcal{U}$. The *motion of a material point* is a smooth section $\varphi : I \to \mathcal{U}$ of the fibration θ, defined on an open interval of time $I \subset \mathcal{T}$, with values in $\mathcal{U}$. The *relative motion* of that material point with respect to the reference frame Φ is the map $p_2 \circ \Phi \circ \varphi : I \to \mathcal{M}$, where $p_2 : \mathcal{T} \times \mathcal{M} \to \mathcal{M}$ is the second projection.

The reference frame $\Phi : \mathcal{U} \to \mathcal{T} \times \mathcal{M}$ is *inertial* if the relative motion of any *free material point* takes place on an *arc of geodesic* of the Riemannian manifold $\mathcal{M}$, at a *constant velocity*.

III. Other possible assumptions. 1. A non-Euclidean Space (3)

Let $\Phi : \mathcal{U} \to \mathcal{T} \times \mathcal{M}$ and $\Psi : \mathcal{U} \to \mathcal{T} \times \mathcal{M}$ be two reference frames. The *change of reference frame* $\Psi \circ \Phi^{-1} : \mathcal{T} \times \mathcal{M} \to \mathcal{T} \times \mathcal{M}$ can be written as

$$(t, m) \mapsto (t, g(t)m)\,, \quad t \in \mathcal{T}\,, \quad m \in \mathcal{M}\,,$$

where $t \mapsto g(t)$ is a smooth map defined on $\mathcal{T}$, with value in the group G.

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where $t \mapsto g(t)$ is a smooth map defined on $\mathcal{T}$, with value in the group G.

Assume that Φ is *inertial*. Then Ψ is inertial *if and only if* for any affinely parametrized geodesic $t \mapsto m(t)$ in $\mathcal{M}$ ($t \in \mathcal{T}$), the curve $t \mapsto g(t)\bigl(m(t)\bigr)$ too is an affinely parametrized geodesic in M.

Since *points* in $\mathcal{M}$ (considered as *constant curves*) are geodesics, a *necessary condition* unless which Ψ cannot be an inertial frame is

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Since *points* in $\mathcal{M}$ (considered as *constant curves*) are geodesics, a *necessary condition* unless which Ψ cannot be an inertial frame is

> for each $m \in \mathcal{M}$, the curve $t \mapsto g(t)m$, $t \in \mathcal{T}$, is an affinely parametrized geodesic in $\mathcal{M}$.

By choosing an origin 0 and a unit of Time, we now identify $\mathcal{T}$ with $\mathbb{R}$. When the above stated necessary condition is satisfied, for each $m \in \mathcal{M}$, the curve $t \mapsto g(t)m$ is an affinely parametrized geodesic in $\mathcal{M}$. Consequently

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- the map $\mathcal{T} \times \mathcal{M} \to \mathcal{M}$, $(t, m) \mapsto g(t) \circ \bigl(g(0)\bigr)^{-1} m$ is the *flow* of a smooth vector field on $\mathcal{M}$, whose integral curves are affinely parametrized geodesics ;

III. Other possible assumptions. 1. A non-Euclidean Space (4)

By choosing an origin 0 and a unit of Time, we now identify $\mathcal{T}$ with $\mathbb{R}$. When the above stated necessary condition is satisfied, for each $m \in \mathcal{M}$, the curve $t \mapsto g(t)m$ is an affinely parametrized geodesic in $\mathcal{M}$. Consequently

- the map $\mathcal{T} \times \mathcal{M} \to \mathcal{M}$, $(t, m) \mapsto g(t) \circ \bigl(g(0)\bigr)^{-1} m$ is the *flow* of a smooth vector field on $\mathcal{M}$, whose integral curves are affinely parametrized geodesics ;
- the map $t \mapsto g(t)\bigl(g(0)\bigr)^{-1}$ is a *one-parameter subgroup* of G. In other terms, there exists $X \in \mathcal{G}$ (the Lie algebra of G) such that

$$g(t)\bigl(g(0)\bigr)^{-1} = \exp(tX)\,, \quad \text{or} \quad g(t) = \exp(tX)g(0)\,.$$

Therefore a necessary condition for the existence of non-constant curves $t \mapsto g(t)$ in G such that $(t,m) \mapsto \bigl(t, g(t)m\bigr)$ is a *change of inertial reference frames*, is the existence of non-zero elements X in the Lie algebra $\mathcal{G}$ of G such that, for any $m \in \mathcal{M}$, the curve $t \mapsto \exp(tX)m$ is an *affinely-parametrized geodesic* in $\mathcal{M}$.

III. Other possible assumptions. 1. A non-Euclidean Space (5)

Therefore a necessary condition for the existence of non-constant curves $t \mapsto g(t)$ in G such that $(t, m) \mapsto \bigl(t, g(t)m\bigr)$ is a *change of inertial reference frames*, is the existence of non-zero elements X in the Lie algebra $\mathcal{G}$ of G such that, for any $m \in \mathcal{M}$, the curve $t \mapsto \exp(tX)m$ is an *affinely-parametrized geodesic* in $\mathcal{M}$.

In the usual case of Classical Mechanics, $\mathcal{M}$ is a 3-dimensional affine Euclidean space and G is the Lie group of affine displacements of $\mathcal{E}$. For any *infinitesimal translation* X of $\mathcal{E}$, the curves $t \mapsto \exp(tX)m$ (where m may be any point in $\mathcal{E}$) are parallel straight lines. We have indeed infinitely many *inertial reference frames* whose relative motions are translations at constant velocities.

III. Other possible assumptions. 1. A non-Euclidean Space (6)

If $\mathcal{M}$ is a three-dimensional sphere S^3, the group G is $\mathbf{SO}(4)$, and if $\mathcal{M}$ is a sheet of a three-dimensional hyperboloid H^3, the group G is the Lorentz group $\mathbf{SO}(3,1)$. In both cases, the above necessary condition for the existence of *non-trivial changes of inertial reference frame* is not fulfillled. Therefore

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> When Space is modelled by a non-Euclidean Riemannian homogeneous manifold of constant non-zero curvature, *there are no non-trivial changes of inertial reference frame*, and *nothing similar to the Galilean group*.

The *Principle of Inertia* describes the way in which a *free material point* evolves in Space-Time. It can be stated as follows.

> There exists on Space-Time a linear connection such that the world line of any free material point is a solution of the second-order differential equation
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> $$\nabla_{\frac{dz(t)}{dt}} \frac{dz(t)}{dt} = 0\,,$$
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> where ∇ is the operator of covariant derivation of that connection.

III. Other possible assumptions. 2. The Kepler connection

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> where ∇ is the operator of covariant derivation of that connection.

The above statement uses only a *local mathematical object* : a linear connection, while other statements of the same Principle presented before used *global mathematical objects* : geodesics, either of the standard fibre of the fibered Space-Time, or of Space-Time itself.

III. Other possible assumptions. 2. The Kepler connection (2)

The very notion of *free material point*, together with the *physical origin of Inertia*, were questioned by *Ernst Mach (1838–1916)* [4]. He did not accept the existence of an *absolute Space* and expressed the idea that the *inertia of a material point* was due to the influence, on that material point, of all other masses in the Universe. For him, *no material point is really free* and the Inertia of a material point is simply *a way to take into account the effects of all other massive objects* whose influence is not explicitly described by a force acting on the considered material point.

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Motivated by *Einstein's Theory of General Relativity*, *Élie Cartan* proposed [1,2] to use connections in *classical Mechanics*, to include not only *inertia*, but also *gravitational forces* in the *geometry of Space-Time*. We describe below a Space-Time endowed with a connection whose geodesics are the Keplerian trajectories of planets in the Solar system.

We take for *Space-Time* $\mathcal{U} = \{(t, x, y, z) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 \neq 0\}$ and for *Time* $\mathcal{T} = \mathbb{R}$ with t as coordinate. The *date map* is $\theta : (t, x, y, z) \mapsto t$. To make notations more convenient, we set $x^0 = t$, $x^1 = x$, $x^2 = y$, $x^3 = z$. Possible values of Latin indices i, j, $\ldots$, will be 1, 2 or 3. We set $r^2 = x^2 + y^2 + z^2$.

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We define on $\mathcal{U}$ a *linear connection*, called the *Kepler connection*, whose *non-zero Christoffel symbols* are

$$\Gamma^i_{0\,0} = -\frac{kx^i}{r^3} \qquad \text{where } k > 0 \text{ is a positive constant}\,.$$

III. Other possible assumptions. 2. The Kepler connection (4)

A parametrized curve $s \mapsto \big(t(s), x(s), y(s), z(s)\big)$ is a *geodesic* of the *Kepler connection* if and only if it is an *integral curve* of the second order differential equation

$$\frac{dt(s)}{ds} = v^0(s)\,, \qquad \frac{dx^i(s)}{ds} = v^i(s)\,,$$

$$\frac{dv^0(s)}{ds} = 0\,, \qquad \frac{dv^i(s)}{ds} = -\frac{k\big(v^0(s)\big)^2 x^i}{r^3}\,.$$

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We see that $v^0(s)$ is a constant v^0. A geodesic will be said *Space-like* if $v^0 = 0$ and *Time-like* if $v^0 \neq 0$.

- *Space-like geodesics* are straight lines contained in hyperplanes $t =$ Constant.
- *Time-like geodesics* are Kepler trajectories of planets, the attractive centre being at $r = 0$.

III. Other possible assumptions. 2. The Kepler connection (5)

The *torsion* of the Kepler connection is zero and the only non-zero components of its *Riemann-Christoffel curvature tensor* are

$$R^k_{i\,0\,0} = -R^k_{0\,i\,0} = -\frac{\partial}{\partial x^i}\left(\frac{kx^k}{r^3}\right), \quad 1 \leq i,\, k \leq 3\,.$$

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Remark The Kepler connection becomes singulal for $r = 0$. It may be possible to build on Space-Time a connection for which collisions with the attractive center are regularized. But at the price of deleting the assumption of the existence of an absolute Time, since regularization of collisions uses as parameter the *Levi-Civita parameter* which is not a smooth function of Time.

We still assume that Space Time is a 4-dimensional manifold $\mathcal{U}$, fibered over *Time* $\mathcal{T}$, which is a real affine 1-dimensional space, by a *date map* $\theta : \mathcal{U} \to \mathcal{T}$.

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A linear connection on $\mathcal{U}$ such that the world lines of free material points (or more generally material points submitted to gravitational forces of distant objects) are affinely parametrized geodesics, *must satisfy the following condition*, which expresses the fact that Time flows at a constant velocity.

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> For any geodesic $s \mapsto z(s)$ in $\mathcal{U}$, the length of the vector $T\theta\left(\dfrac{dz(s)}{ds}\right)$ does not depend on s.

Indeed, the existence of an *affine structure* on $\mathcal{T}$ allows us to compare the lengths of vectors tangent to $\mathcal{T}$ at different points.

In terms of the *Christoffel symbols* associated with local coordinates $(x^0 = t, x^1, x^2, x^3)$ on $\mathcal{U}$, $x^0 = t$ being the time, the above condition is

$$\Gamma^0_{i\,j} = 0 \quad \text{for all } i, j, \quad 0 \leq i, j \leq 3\,.$$

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$$\Gamma^0_{i\,j} = 0 \quad \text{for all } i, j, \quad 0 \leq i, j \leq 3\,.$$

An *equivalent*, but *more geometric way* of expressing this condition uses the fact that the tangent space $T\mathcal{U}$ is *foliated into codimension-1 leaves*, each leaf being the set of vectors $w \in T\mathcal{U}$ such that $T\theta(w)$ is a vector tangent to $\mathcal{T}$ of a *given length*. The above condition amounts to say that *the geodesic flow preseves this foliation*, or in other words that for each $w \in T\mathcal{U}$, the subset C_w of horizontal vectors tangent to $T\mathcal{U}$ at w is *tangent to the leaf containing w*.

III. Other possible assumptions. 4. The Lagrange 2-form

In his book *Structure des systèmes dynamiques* [7], *J. M Souriau* defines and uses the *Space of Evolution* of a mechanical system. For a material point evolving in *Leibniz Space-Time* $\mathcal{L}$, it is the 7-dimensional space made by all the possible values of $(t, x, \overrightarrow{v})$, where

- $t \in \mathcal{T}$ is the time,
- $x \in \mathcal{E}$ is the position the material point observed in a given inertial reference frame $\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E}$,
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The *Lagrange 2-form* on this Space of Evolution is (m being the *mass* of the particle, and $\overrightarrow{f} \in \overrightarrow{\mathcal{E}}$ the *force* which acts on it, assumed to be a given function of $(t, x, \overrightarrow{v})$)

$$\omega_\Phi = (m\overrightarrow{dv} - \overrightarrow{f}\,dt) \wedge (\overrightarrow{dx} - \overrightarrow{v}\,dt) = \sum_{i=1}^{3} (m dv^i - f^i\,dt) \wedge (dx^i - v^i\,dt)\,.$$

III. Other possible assumptions. 4. The Lagrange 2-form (2)

The Space of Evolution of a more general mechanical system with n degrees of freedom is a $(2n+1)$-dimensional manifold on which one can define a *Lagrange 2-form* ω. When the system is *regular* the *rank* of ω is $2n$ and the lines which describe the *time evolution* of the system are the integral curves of the rank-1 distribution $\ker \omega$.

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Let us come back to the Space of Evolution of a material point and to its Lagrange 2-form. The above definitions are given in terms of a given *inertial reference frame*. However, these definitions can be adapted to *avoid the use of any reference frame*. Once a *unit of Time* has been chosen, we can define the sub-bundle of the tangent bundle to Space-Time

$$T^1\mathcal{L} = \{ w \in T\mathcal{L} \mid T\theta(w) = 1 \} \,.$$

III. Other possible assumptions. 4. The Lagrange 2-form (3)

For each inertial reference frame $\Phi : \mathcal{L} \to \mathcal{T} \times \mathcal{E}$, the map $T\Phi : T\mathcal{L} \to T(\mathcal{T} \times \mathcal{E})$, *restricted to* $T^1\mathcal{L}$ is a diffeomorphism Ψ from $T^1\mathcal{L}$ onto the Space of Evolution defined with the use of the reference frame Φ. We can therefore take the pull-back $\omega = \Psi^*\omega_\Phi$ of the Lagrange 2-form ω_Φ defined with the use of Φ. The 2-form ω *does not depend on the choice of the inertial reference frame* Φ.

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The *Space of Evolution* of our material point, equipped with its *Lagrange 2-form*, $(T^1\mathcal{L}, \omega)$, is now well defined, independently of any choice of a particular reference frame.

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The *Space of Evolution* of our material point, equipped with its *Lagrange* 2*-form*, $(T^1\mathcal{L}, \omega)$, is now well defined, independently of any choice of a particular reference frame.

Souriau calls *Principle of Maxwell* the property

$$d\omega = 0\,.$$

It imposes strong restrictions on the way in which the force $\overrightarrow{f}$ depends on $(t, x, \overrightarrow{v})$.

Summary and conclusion

- In Classical Mechanics, *Time* has an *absolute character* and is mathematically modelled by a real affine one-dimensional space $\mathcal{T}$. *Space* is mathematically modelled by a real, four-dimensional manifold $\mathcal{L}$, fibered on $\mathcal{T}$ by a *date map* $\theta : \mathcal{L} \to \mathcal{T}$.

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- For each $t \in \mathcal{T}$, $\mathcal{E}_t = \theta^{-1}(t)$ is the *Space at time* t. Once a unit of length has been chosen, it has the structure of a real three-dimensional affine Euclidean space.

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- For each $t \in \mathcal{T}$, $\mathcal{E}_t = \theta^{-1}(t)$ is the *Space at time* t. Once a unit of length has been chosen, it has the structure of a real three-dimensional affine Euclidean space.
- The *Principle of Inertia* can be expressed as follows : *There exists on Space-Time $\mathcal{L}$ an affine space structure such that the date map $\theta : \mathcal{L} \to \mathcal{T}$ is an affine map and that the motion of any free material point is an affine section of the fibration θ.*

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- For each $t \in \mathcal{T}$, $\mathcal{E}_t = \theta^{-1}(t)$ is the *Space at time* t. Once a unit of length has been chosen, it has the structure of a real three-dimensional affine Euclidean space.
- The *Principle of Inertia* can be expressed as follows : *There exists on Space-Time $\mathcal{L}$ an affine space structure such that the date map $\theta : \mathcal{L} \to \mathcal{T}$ is an affine map and that the motion of any free material point is an affine section of the fibration θ*.
- Let $\mathcal{E}$ be the *standard fibre* of the fibered Space-Time $\theta : \mathcal{L} \to \mathcal{T}$. A *reference frame* is a global trivialization $\Phi : \mathcal{L} \to \mathcal{E}$ of the ifibered Space-Time $\mathcal{L}$. The reference frame Φ is *inertial* (or *Galilean*) if it is an *affine map*.

Summary and conclusion (2)

- The *Galilean group* is the *group of changes of Galilean reference frames* plus the *translations in Time*.

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- If we assume that *Space is curved*, i.e., that instead of being an affine Euclidean space, the standard fibre $\mathcal{E}$ of the fibration $\theta : \mathcal{L} \to \mathcal{T}$ is an *homogeneous Riemannian space of constant non-zero curvature*, (a *Sphere* S^3, homogeneous space of $SO(4)$, or a *sheet of hyperboloid* H^3, homogeneous space of $SO(3,1)$), there is *no analogue* of the Galilean group : the inertial reference frame is *essentially unique*, up to a fixed change of frame in Space.

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- As observed by É. Cartan, instead of a global structure (such as an affine structure), on can use a *local structure*, such as a connection, for the formulation of the Principle of Inertia. *Gravitational forces* can be included in the geometry of Space-Time. We gave an example of a linear connection whose geodesics are models of the Keplerian orbits of planets.

Summary and conclusion (3)

- The existence of a *date map* imposes strong restrictions on the linear connection on Space-Time used for the mathematical formulation of the Principle of Inertia.

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- Once global assumptions on Space have been deleted, it seems natural to question the validity of the assumption about the existence of an *absolute Time*. If we delete this very strong global assumption, we no more have a date map, and enter the realm of *Relativity theories*.

Summary and conclusion (3)

- The existence of a *date map* imposes strong restrictions on the linear connection on Space-Time used for the mathematical formulation of the Principle of Inertia.
- Once global assumptions on Space have been deleted, it seems natural to question the validity of the assumption about the existence of an *absolute Time*. If we delete this very strong global assumption, we no more have a date map, and enter the realm of *Relativity theories*.
- In Classical Mechanics, the Lagrangian and Hamiltonian formalisms are linked to the use of an *inertial frame*. The notions of *Space of Evolution* and of 2*-form of Lagrange* can be expressed in Space-Time, without any choice of a particular reference frame.

Thanks

It was a pleasure and a honour for me to participate to this conference in honour of my colleague and friend Franco Magri, and to listen to the very nice lectures of all the other participants. My warmest thanks to the organizers, Professor G. Falqui, Professor B. Konopelchenko and Professor M. Pedroni for their generous and friendly hospitality.

Thanks

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Happy birthday, Professor Magri, and many more happy years with a lot of beautiful scientific achievements !

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