Dirac brackets and bihamiltonian structures

Charles-Michel Marle

cmm1934@orange.fr

Université Pierre et Marie Curie
Paris, France
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Thanks
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In what follows I will describe Dirac’s theory of generalized Hamiltonian dynamics, and I will consider its links with the theory of bihamiltonian systems.
We consider a mechanical system with a smooth manifold $Q$ as configuration space. The dynamical properties of the system are described by a smooth Lagrangian $L : TQ \rightarrow \mathbb{R}$. Possible motions of the system are curves $t \mapsto q(t)$, parametrized by the time $t$, defined on intervals $[t_0, t_1] \subset \mathbb{R}$, which are extremals of the action integral

\[ I = \int_{t_0}^{t_1} L \left( \frac{dq(t)}{dt} \right) \, dt, \]

with fixed endpoints.
Lagrangian formalism and Legendre map (1)

The curve \( t \mapsto q(t) \) is an extremal of the action integral if and only if it satisfies Lagrange equations, which, in a chart of \( Q \) with local coordinates \( (q^1, \ldots, q^n) \), and the associated chart of \( TQ \) with local coordinates \( (q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) \), are

\[
\frac{d}{dt} \left( \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \right) - \frac{\partial L(q, \dot{q})}{\partial q^i} = 0, \quad 1 \leq i \leq n.
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\]

When the \textit{Legendre map}

\[ \mathcal{L} : TQ \to T^*Q, \quad (q, \dot{q}) \mapsto (q, p) \text{ with } p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \]

is a (local) diffeomorphism, one may (locally) define a Hamiltonian \( H \) on \( T^*Q \) by setting

\[
H(q, p) = \sum_{i=1}^{n} \dot{q}^i p_i - L(q, \dot{q}),
\]
where $\dot{q}^i$ and $(q, \dot{q}) = \mathcal{L}^{-1}(q, p)$ are expressed in terms of $(q, p)$ by means of the (local) inverse $\mathcal{L}^{-1}$ of the Legendre map. Under these assumptions, Lagrange equations are (locally) equivalent to Hamilton equations,

$$
\frac{dq^i}{dt} = \frac{\partial H(q, p)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(q, p)}{\partial q^i}.
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Lagrangian formalism and Legendre map (2)

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\]

When the Legendre map is not a (local) diffeomorphism, we still can define a Hamiltonian on \( TQ \oplus T^*Q \) by

\[
H(q, \dot{q}, p) = \sum_{i=1}^{n} \dot{q}^i p_i - L(q, \dot{q}).
\]

That Hamiltonian is not a function defined on \( T^*Q \).
Primary constraints (1)

Dirac does not assume that the Legendre map is a local diffeomorphism. Instead, he (tacitly) assumes that it is a map of constant rank $2n - r$, with $1 \leq r \leq n$. Let $D_0 = \mathcal{L}(TQ)$ be its image. Dirac assumes that there exist $r$ smooth functions $\Phi_\alpha : T^*Q \to \mathbb{R}$, $1 \leq \alpha \leq r$, such that $D_0$ is defined by the equations

$$\Phi_\alpha = 0, \quad 1 \leq \alpha \leq r,$$

the $d\Phi_\alpha$ being linearly independent on $D_0$. These equations are called primary constraints. The functions $\Phi_\alpha$ are called the primary constraint functions.
Primary constraints (2)

These assumptions are valid locally: since $\mathcal{L}$ is of constant rank $2n - r$, each point in $TQ$ has an open neighbourhood $U$ in $TQ$ such that $\mathcal{L}(U)$ is a smooth $(2n - r)$-dimensional submanifold of $T^*Q$ defined by equations of that form, the $\Phi_i$ being smooth functions defined on some open subset $V$ of $T^*Q$ containing $\mathcal{L}(U)$. 
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The partial derivatives of $H : TQ \oplus T^*Q \to \mathbb{R}$ are

$$
\frac{\partial H(q, \dot{q}, p)}{\partial q^i} = -\frac{\partial L(q, \dot{q})}{\partial q^i}, \quad \frac{\partial H(q, \dot{q}, p)}{\partial \dot{q}^i} = p_i - \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i},
$$

$$
\frac{\partial H(q, \dot{q}, p)}{\partial p_i} = \dot{q}^i.
$$
Generalized Hamiltonian dynamics (1)

Let

\[ \Gamma_L = \{(q, \dot{q}, p) \in TQ \oplus T^*Q; (q, p) = \mathcal{L}(q, \dot{q})\} \]

be the graph of the Legendre map. We see that

\[ \frac{\partial H(q, \dot{q}, p)}{\partial \dot{q}^i} = 0 \quad \text{when} \quad (q, \dot{q}, p) \in \Gamma_L. \]
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In other words, on the graph $\Gamma_L$ (which is a $2n$-dimensional submanifold of the $3n$-dimensional manifold $TQ \oplus T^*Q$), the Hamiltonian $H$ does not depend on the variables $\dot{q}^i$. 
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In other words, on the graph \( \Gamma_L \) (which is a \( 2n \)-dimensional submanifold of the \( 3n \)-dimensional manifold \( TQ \oplus T^*Q \)), the Hamiltonian \( H \) does not depend on the variables \( \dot{q}^i \).

Dirac considers that there exists a smooth function \( \hat{H} \), defined on an open subset of \( T^*Q \) containing \( D_0 \), such that

\[ H(q, \dot{q}, p) = \hat{H}(q, p) \quad \text{when } (q, \dot{q}, p) \in \Gamma_L. \]
The function $\hat{H}$ with these properties, when it exists, is not unique: we may add to it any smooth function which vanishes on the image $D_0$ of the Legendre map.
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For the existence of $\hat{H}$, we may have to restrict the Legendre map $\mathcal{L}$ to a suitable open subset of $TQ$, such that for each point $(q, p) \in D_0$, the intersection of that open subset with $\mathcal{L}^{-1}(q, p)$ is connected. Following Dirac we will assume that functions with these properties do exist, and we choose arbitrarily one of them.
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Instead of $\hat{H}$ and $\Phi_\alpha$, defined on $T^*Q$, it is more convenient to consider the functions, defined on $TQ \oplus T^*Q$,

$$\tilde{H} = \hat{H} \circ \pi_{T^*Q}, \quad \tilde{\Phi}_\alpha = \Phi_\alpha \circ \pi_{T^*Q}, \quad 1 \leq \alpha \leq r,$$

$$\pi_{T^*Q} : TQ \oplus T^*Q \to T^*Q$$

being the canonical submersion.
Both $H$ and $\tilde{H}$ are functions defined on $TQ \oplus T^*Q$, which are equal on the graph $\Gamma_L$ of $\mathcal{L}$:

$$H(q, \dot{q}, p) = \tilde{H}(q, p) = \tilde{H}(q, \dot{q}, p) \quad \text{when} \quad (q, p) = \mathcal{L}(q, \dot{q}) .$$
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So for each $(q, \dot{q}, p) \in \Gamma_L$, $dH(q, \dot{q}, p)$ and $d\tilde{H}(q, \dot{q}, p)$ are equal on the subspace of vectors tangent to $\Gamma_L$. Moreover,

$$\bigcap_{\alpha=1}^{r} \ker d\tilde{\Phi}_\alpha(q, \dot{q}, p) \subset \ker d(H - \tilde{H})(q, \dot{q}, p).$$
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The theory of Lagrange multipliers shows that, for $(q, \dot{q}, p) \in \Gamma_L$,

$$dH(q, \dot{q}, p) = d\tilde{H}(q, \dot{q}, p) + \sum_{\alpha=1}^{r} v^\alpha d\tilde{\Phi}_\alpha(q, \dot{q}, p).$$
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Generalized Hamiltonian dynamics (4)

For each \( (q, \dot{q}, p) \in \Gamma_L \), the family of Lagrange multipliers \((v^\alpha, 1 \leq \alpha \leq r)\) depends on that point and of the time \(t\), and may not be unique.
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\[
\frac{\partial H(q, \dot{q}, p)}{\partial q^i} = -\frac{\partial L(q, \dot{q})}{\partial q^i}, \quad \frac{\partial H(q, \dot{q}, p)}{\partial \dot{q}^i} = 0, \quad \frac{\partial H(q, \dot{q}, p)}{\partial p_i} = \dot{q}^i.
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\]

Therefore we have, for \((q, p) = \mathcal{L}(q, \dot{q})\),

\[
\frac{\partial \hat{H}(q, p)}{\partial q^i} = \frac{\partial \hat{H}(q, \dot{q}, p)}{\partial q^i} = -\frac{\partial L(q, \dot{q})}{\partial q^i} - \sum_{\alpha=1}^{r} v^\alpha(q, \dot{q}, t) \frac{\partial \tilde{\Phi}_\alpha(q, \dot{q}, p)}{\partial q^i},
\]

\[
\frac{\partial \hat{H}(q, p)}{\partial p_i} = \frac{\partial \hat{H}(q, \dot{q}, p)}{\partial p_i} = \dot{q}^i - \sum_{\alpha=1}^{r} v^\alpha(q, \dot{q}, t) \frac{\partial \tilde{\Phi}_\alpha(q, \dot{q}, p)}{\partial p_i}.
\]
Since $\tilde{\Phi}_\alpha = \Phi_\alpha \circ \pi_{T^*Q}$, these equations become, for $(q, p) = \mathcal{L}(q, \dot{q})$,

$$\frac{\partial \hat{H}(q, p)}{\partial q^i} = -\frac{\partial L(q, \dot{q})}{\partial q^i} - \sum_{\alpha=1}^{r} v^\alpha(q, \dot{q}, t) \frac{\partial \Phi_\alpha(q, p)}{\partial q^i},$$

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\]

From the equations of motion in Lagrange’s formalism

\[
\frac{dq^i}{dt} = \dot{q}^i, \quad \frac{d}{dt} \left( \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i} \right) = \frac{\partial L(q, \dot{q})}{\partial q^i}
\]

we deduce the equations of motion in the generalized Hamilton’s formalism:
Generalized Hamiltonian dynamics (6)

\[
\begin{align*}
\frac{dq^i}{dt} &= \frac{\partial \hat{H}(q, p)}{\partial p_i} + \sum_{\alpha=1}^{r} v^\alpha(q, \dot{q}, t) \frac{\partial \Phi_\alpha(q, p)}{\partial p_i}, \\
\frac{dp_i}{dt} &= -\frac{\partial \hat{H}(q, p)}{\partial q^i} + \sum_{\alpha=1}^{r} v^\alpha(q, \dot{q}, t) \frac{\partial \Phi_\alpha(q, p)}{\partial q^i}.
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\end{align*}
\]

These equations, valid for \((q, p) = \mathcal{L}(q, \dot{q})\), follow from Lagrange’s equations. Under Dirac’s assumptions (\(\mathcal{L}\) of constant rank, existence of the functions \(\hat{H}\) and \(\Phi_\alpha\)), once these (not uniquely determined) functions are chosen, for each solution \(t \mapsto q(t)\) of Lagrange’s equations, \(t \mapsto (q(t), \dot{q}(t), p(t))\) is a solution of the above generalized Hamilton’s equations (for a suitable choice of the \(v^\alpha(q, \dot{q}, t)\), which may not be unique). We have set \(\dot{q}(t) = \frac{dq(t)}{dt}\) and \((q(t), p(t)) = \mathcal{L}(q(t), \dot{q}(t))\).
Secondary constraints (1)

The generalized Hamilton’s equations

\[ \begin{align*}
\frac{dq^i}{dt} &= \frac{\partial \hat{H}(q, p)}{\partial p_i} + \sum_{\alpha=1}^{r} v^\alpha \frac{\partial \Phi_\alpha(q, p)}{\partial p_i}, \\
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\end{align*} \]

can be considered as differential equations on $T^*Q$ on their own, the $v^\alpha$ being now considered as unknown functions of the time $t$. From that point of view, they make an under-determined system, since the $v^\alpha$ can be chosen arbitrarily.
Secondary constraints (2)

Using the Poisson bracket on $T^*Q$ associated to its canonical symplectic structure, these equations can be written

$$\frac{dg}{dt} = \{ \hat{H}, g \} + \sum_{\alpha=1}^{r} v^\alpha \{ \Phi_\alpha, g \},$$

where $g$ is any smooth function on $T^*Q$. Hamilton’s generalized equations in local coordinates are obtained when $g$ is one of the local coordinate functions $q^i$ or $p_i$. 
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where $g$ is any smooth function on $T^*Q$. Hamilton’s generalized equations in local coordinates are obtained when $g$ is one of the local coordinate functions $q^i$ or $p_i$. To be an image, by the Legendre map $\mathcal{L}$, of a solution of Lagrange’s equations, a solution $t \mapsto (q(t), p(t))$ of the generalized Hamilton’s equations must lie in $D_0 = \mathcal{L}(TQ)$. i.e., must satisfy

$$\Phi_\alpha(q(t), p(t)) = 0 \quad \text{for all } t, \quad 1 \leq \alpha \leq r.$$
Secondary constraints (3)

That necessary condition is satisfied if the starting point \((q(t_0), p(t_0))\) of that solution is in \(D_0\) and if the following compatibility conditions are satisfied:

\[
\frac{d\Phi_\beta}{dt} = \{\hat{H}, \Phi_\beta\} + \sum_{\alpha=1}^{r} v^\alpha \{\Phi_\alpha, \Phi_\beta\} = 0, \quad 1 \leq \beta \leq r.
\]

When explicited, these equations may give rise to:
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- impossible equalities such as \(1 = 0\);
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- equalities satisfied if \(\Phi_\alpha = 0\) for \(1 \leq \alpha \leq r\);
- impossible equalities such as \(1 = 0\);
- equations which restrict the generality of the \(v^\alpha\).
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When explicited, these equations may give rise to:
- equalities satisfied if $\Phi_\alpha = 0$ for $1 \leq \alpha \leq r$;
- impossible equalities such as $1 = 0$;
- equations which restrict the generality of the $v^\alpha$;
- new equalities, called secondary constraints $\chi_k(p, q) = 0$, $k = 1, 2, \ldots$.

the $\chi_k$ being smooth functions on open subsets of $T^*Q$.  

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Unless impossible equalities occur (which means that the Lagrangian used is inconsistent), secondary constraints lead to new *compatibility conditions*

\[
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which again may give rise to satisfied equalities, impossible equalities, new relations involving the \(v^\alpha\) and new secondary constraints.
Dirac’s algorithm (1)

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which again may give rise to satisfied equalities, impossible equalities, new relations involving the \( v^\alpha \) and new secondary constraints.

The process, called *Dirac’s algorithm*, is pursued until no new constraints appear.
Dirac's algorithm (2)

Assuming that no impossible equation occurred, we are left with a total of $s$ secondary constraints

$$\chi_k(q, p) = 0, \quad 1 \leq k \leq s,$$

and, eventually, $u$ nonhomogeneous linear equations, with functions on open subsets of $T^*Q$ as coefficients, which must be satisfied by the $v^{\alpha}$:

$$\sum_{\alpha=1}^{r} A^{l}_{\alpha}(q, p)v^{\alpha} = B^{l}(q, p), \quad 1 \leq l \leq u.$$
Finally we have a total of $r + s$ constraints

$$\Phi_\alpha(q, p) = 0, \ 1 \leq \alpha \leq r, \quad \chi_k(q, p) = 0, \ 1 \leq k \leq s,$$

which define a subset $D_f$ of $T^*Q$, contained in the image $D_0$ of the Legendre map $L$. The functions $\Phi_\alpha$ and $\chi_k$ are called the primary and secondary constraint functions. We will assume that their differentials are linearly independent at each point in $D_f$. So $D_f$ is a smooth $(2n - r - s)$-dimensional submanifold of $T^*Q$, called the final constraint submanifold, contained in the initial constraint submanifold $D_0$. 

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Dirac brackets and bihamiltonian systems – p. 21/46
Finally we have a total of \( r + s \) constraints

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which define a subset \( D_f \) of \( T^*Q \), contained in the image \( D_0 \) of the Legendre map \( \mathcal{L} \). The functions \( \Phi_\alpha \) and \( \chi_k \) are called the primary and secondary constraint functions. We will assume that their differentials are linearly independent at each point in \( D_f \). So \( D_f \) is a smooth \((2n - r - s)\)-dimensional submanifold of \( T^*Q \), called the final constraint submanifold, contained in the initial constraint submanifold \( D_0 \).

Only the primary constraint functions \( \Phi_\alpha \) appear in the generalized Hamilton equations

\[
\frac{dg}{dt} = \{ \widehat{H}, g \} + \sum_{\alpha=1}^{r} v^\alpha \{ \Phi_\alpha, g \}, \quad g \text{ any smooth function on } T^*Q.
\]
Geometrical Interpretation Let $X_{\hat{H}}$ and $X_{\Phi_\alpha}$ be the Hamiltonian vector fields on $T^*Q$ with Hamiltonians $\hat{H}$ and $\Phi_\alpha$, $(1 \leq \alpha \leq r)$, respectively.
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Geometrical Interpretation

Let $X_{\widehat{H}}$ and $X_{\Phi_{\alpha}}$ be the Hamiltonian vector fields on $T^*Q$ with Hamiltonians $\widehat{H}$ and $\Phi_{\alpha}$, $(1 \leq \alpha \leq r)$, respectively.

Generally speaking, $X_{\widehat{H}}$ is not tangent to the initial constraint submanifold $D_0$. Dirac’s algorithm solves the following problem:

Find a submanifold $D_f \subset D_0$ such that for each point $z$ of that submanifold, there exist coefficients $v^\alpha$ for which the vector

$$X_{\widehat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_{\alpha}}(z)$$

is tangent to $D_f$ at $z$. 
In other words, for each $z \in D_f$, the intersection of $T_z D_f$ with the affine subspace of $T_z (T^*Q)$ made by the vectors $X_{\hat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z)$, for all reals $v^\alpha$, must be not empty.
In other words, for each $z \in D_f$, the intersection of $T_z D_f$ with the affine subspace of $T_z (T^* Q)$ made by the vectors $X_{\hat{H}} (z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha} (z)$, for all reals $v^\alpha$, must be not empty.

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Final constraint submanifold (3)
In other words, for each $z \in D_f$, the intersection of $T_z D_f$ with the affine subspace of $T_z (T^* Q)$ made by the vectors $X \tilde{H}(z) + \sum_{\alpha=1}^{r} v^\alpha X \Phi_\alpha (z)$, for all reals $v^\alpha$, must be not empty.

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For example, if one of the vectors $X \Phi_\alpha (z)$ is tangent to $D_f$, the corresponding coefficient $v^\alpha$ can be any real number and that intersection contains an affine straight line parallel to that vector.
In other words, for each \( z \in D_f \), the intersection of \( T_z D_f \) with the affine subspace of \( T_z (T^*Q) \) made by the vectors \( X_{\hat{H}}(z) + \sum_{\alpha=1}^{r} v^\alpha X_{\Phi_\alpha}(z) \), for all reals \( v^\alpha \), must be not empty.

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These considerations lead Dirac to distinguish two kinds of constraints: first class constraints and second class constraints.
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For example, if one of the vectors \( X_{\Phi_\alpha}(z) \) is tangent to \( D_f \), the corresponding coefficient \( v^\alpha \) can be any real number and that intersection contains an affine straight line parallel to that vector.

These considerations lead Dirac to distinguish two kinds of constraints: \textit{first class constraints} and \textit{second class constraints}.

The distinction between \textit{first class} and \textit{second class} constraints is independent of the distinction between \textit{primary} and \textit{secondary} constraints.
First and second class constraints (1)

Definition  A smooth function on $T^*Q$ is said to be first class if its Poisson bracket with the constraint functions (primary as well as secondary) $\Phi_\alpha$ and $\chi_k$ vanishes on $D_f$, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be second class.
First and second class constraints (1)

**Definition**  A smooth function on $T^*Q$ is said to be *first class* if its Poisson bracket with the constraint functions (primary as well as secondary) $\Phi_\alpha$ and $\chi_k$ vanishes on $D_f$, $1 \leq \alpha \leq r$, $1 \leq k \leq s$.

A function which is not first class is said to be *second class*. These definitions apply to the constraint functions $\Phi_\alpha$ and $\chi_k$ themselves. So we distinguish between *first class* and *second class constraint functions*. 
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The Poisson bracket of two first class functions is first class (Jacobi identity). Any linear combination of first class functions, with functions defined on $T^*Q$ as coefficients, is first class.
First and second class constraints (1)

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The Poisson bracket of two first class functions is first class (Jacobi identity). Any linear combination of first class functions, with functions defined on $T^*Q$ as coefficients, is first class.

**Geometrical interpretation**. A smooth function $\Psi$ is first class if and only if the Hamiltonian vector field $X_\Psi$ is everywhere tangent to the final constraint submanifold $D_f$. 

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When the constraint function $\Psi$ is chosen among the $\Phi_\alpha$'s and $\chi_k$'s obtained with Dirac's algorithm, the corresponding Hamiltonian vector field $X_\psi$ may be tangent to the final constraint submanifold $D_f$ at some points, but not everywhere. For that reason, instead of the original constraint functions $\Phi_\alpha$ and $\chi_k$, Dirac uses a set of $r + s$ new constraint functions $\Psi_\gamma$, $1 \leq \gamma \leq r + s$, obtained from the original ones by a linear transformation whose coefficients are functions on $T^*Q$, the determinant of that linear transformation being nowhere zero. The submanifold $D_f \subset T^*Q$ can now be defined by the equations

$$\Theta_\gamma(q, p) = 0, \quad 1 \leq \gamma \leq r + s,$$

instead of $\Psi_\alpha(q, p) = 0$ and $\chi_k(q, p) = 0$, $1 \leq \alpha \leq r$, $1 \leq k \leq s$. 

First and second class constraints (2)
Dirac chooses the transformation which yields the $\Theta_{\gamma}$’s as linear combinations of the $\Phi_{\alpha}$’s and the $\chi_k$’s, in such a way that the number of first class $\Theta_{\gamma}$’s is the largest possible. By reordering, we may assume that the $\Theta_{\gamma}$ are:

*first class for* $1 \leq \gamma \leq k$,  
*second class for* $k + 1 \leq \gamma \leq r + s$. 

**Poisson brackets of second class constraints**
Dirac chooses the transformation which yields the $\Theta_\gamma$’s as linear combinations of the $\Phi_\alpha$’s and the $\chi_k$’s, in such a way that the number of first class $\Theta_\gamma$’s is the largest possible. By reordering, we may assume that the $\Theta_\gamma$ are:

*first class for* $1 \leq \gamma \leq k$, *second class for* $k + 1 \leq \gamma \leq r + s$.

Dirac proves that the matrix whose coefficients are the Poisson brackets of pairs of second class $\Theta_\gamma$’s,

$$\{\Theta_\alpha, \Theta_\beta\}, \quad k + 1 \leq \alpha, \beta \leq r + s$$

is invertible at each point of $D_f$. The Poisson bracket being skew-symmetric, it implies that the number $r + s - k$ of second class constraints is even. We will set $r + s - k = 2p$.

This result has the following symplectic explanation.
For each point $z \in D_f$ the tangent space $T_z D_f$ is the annihilator of the vector subspace of $T^*_z (T^* Q)$ generated by the $d \Theta_\gamma (z)$, $1 \leq \gamma \leq r + s$. Equipped with $\omega (z)$, $T_z (T^* Q)$ is a symplectic vector space. The symplectic orthogonal $\text{orth}(T_z D_f)$ of its vector subspace $T_z D_f$ is the vector subspace of $T_z (T^* Q)$ generated by the Hamiltonian vectors $X_{\Theta_\gamma} (z)$, $1 \leq \gamma \leq r + s$. 
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When $\Theta_\gamma$ is first class, $X_{\Theta_\gamma}(z) \in T_z D_f \cap \text{orth}(T_z D_f)$. This subspace is the common kernel of the 2-forms induced by $\omega(z)$ both on the vector subspace $T_z D_f$ and on its symplectic orthogonal $\text{orth}(T_z D_f)$.
For each point $z \in D_f$ the tangent space $T_z D_f$ is the annihilator of the vector subspace of $T^*_z(T^*Q)$ generated by the $d\Theta_\gamma(z)$, $1 \leq \gamma \leq r + s$. Equipped with $\omega(z)$, $T_z(T^*Q)$ is a symplectic vector space. The symplectic orthogonal $\text{orth}(T_z D_f)$ of its vector subspace $T_z D_f$ is the vector subspace of $T_z(T^*Q)$ generated by the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, $1 \leq \gamma \leq r + s$.

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It seems that Dirac tacitly assumes that the 2-form induced on $D_f$ by $\omega$ is of constant rank. I tried (without success) to prove that the constancy of rank followed from Dirac’s algorithm.
Symplectic explanation (2)

Under this assumption, we can indeed *split (at least locally)* the constraint functions $\Theta_\gamma$ into first and second class. At each point $z \in D_f$, the Hamiltonian vectors $X_{\Theta_\gamma}(z)$, with $\Theta_\gamma$ second class, form a basis of a symplectic vector subspace of $T_z(T^*Q)$. That explains why the matrix of Poisson brackets of secondary constraint functions is invertible at each point $z \in D_f$. 
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In what follows we will denote by $\Theta_\gamma$ ($1 \leq \gamma \leq k$) the first class constraint functions and by $\Psi_\delta = \Theta_{k+\delta}$ ($1 \leq \delta \leq 2p$) the second class constraint functions.
The Dirac bracket (1)

Dirac introduces, for functions defined on $T^*Q$, a modified Poisson bracket (we will call it the *Dirac bracket* and denote it by $\{ , \}_D$) with the following properties:
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- that modified Poisson bracket is defined on the open neighbourhood $W$ of $D_f$ in $T^*Q$ on which the matrix with coefficients $\{ \Psi_\gamma, \Psi_\delta \}$ is invertible;
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1. That modified Poisson bracket is defined on the open neighbourhood $W$ of $D_f$ in $T^*Q$ on which the matrix with coefficients $\{\Psi_\gamma, \Psi_\delta\}$ is invertible;

2. The Dirac bracket $\{\Psi_\gamma, g\}_D$ of a second class constraint function $\Psi_\gamma (1 \leq \gamma \leq 2p)$ with any smooth function $g$ vanishes identically on $W$;
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- that modified Poisson bracket is defined on the open neighbourhood $W$ of $D_f$ in $T^*Q$ on which the matrix with coefficients $\{ \Psi_\gamma, \Psi_\delta \}$ is invertible;
- the Dirac bracket $\{ \Psi_\gamma, g \}_D$ of a second class contraint function $\Psi_\gamma$ (1 ≤ $\gamma$ ≤ 2p) with any smooth function $g$ vanishes identically on $W$;
- When $f$ and $g$ are two smooth functions on $T^*Q$ and $z \in W$ a point at which $\{ f, \Psi_\gamma \} = 0$ and $\{ g, \Psi_\gamma \} = 0$, 1 ≤ $\gamma$ ≤ 2p, then

$$
\{ f, g \}_D(z) = \{ f, g \}(z) .
$$
The Dirac bracket (2)

We have seen that the matrix with coefficients

\[ M_{\alpha \beta} = \{ \Psi_{\alpha}, \Psi_{\beta} \}, \quad 1 \leq \alpha, \beta \leq 2p \]

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is invertible. Let \( C_{\alpha\beta} \) be the coefficients of its inverse. They are smooth functions on \( W \) such that

\[ \sum_{\beta=1}^{2p} M_{\alpha\beta} C_{\beta\gamma} = \delta_{\alpha\gamma}. \]
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The Dirac bracket of two smooth functions \( f \) and \( g \) is defined as

\[ \{ f, g \}_D = \{ f, g \} - \sum_{\alpha=1}^{2p} \sum_{\beta=1}^{2p} \{ f, \Psi_\alpha \} C_{\alpha\beta} \{ \Psi_\beta, g \}. \]
The Dirac bracket (3)

In [1] Dirac proves, by direct calculations, that his bracket has all the properties of a Poisson bracket (skew-symmetry, Leibniz identity and Jacobi identity).
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$$\frac{dg}{dt} = \{\hat{H} + \sum_{\alpha=1}^{r} v^\alpha \Phi_\alpha, g\} = \{H_T, g\}.$$
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The function $H_T = \hat{H} + \sum_{\alpha=1}^{r} v^\alpha \Phi_\alpha$ is first class, so the generalized Hamilton equation can be written with the Dirac bracket, as well as with the ordinary bracket

$$\frac{dg}{dt} = \{ H_T, g \}_D .$$
The Poisson-Dirac bivector (1)

The usual Poisson bracket is built with the bivector field $\Lambda$ on $T^*Q$, given in Darboux coordinates by

$$\Lambda = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i}.$$

Let $\Lambda^\#: T^*Q \rightarrow TQ$ be the bundle morphism determined by $\Lambda$. For smooth functions $f$ and $g$,

$$\{f, g\} = \Lambda(df, dg) = i(X_f)dg, \quad \text{with } X_f = \Lambda^\#(df).$$
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Let $\Lambda_D$ be the bivector field corresponding to $\{ , \}_D$.

Let $F$ be the rank $2p$ vector subbundle of $T(T^*Q)$ generated by the Hamiltonian vector fields $X_{\Psi_\alpha}, 1 \leq \alpha \leq 2p$, and $G = \text{orth } F$ be its symplectic orthogonal. Both $F$ and $G$ are symplectic vector subbundles and

$$T(T^*Q) = F \oplus \text{orth } F = F \oplus G.$$
The Poisson-Dirac bivector (2)

By duality $T^*(T^*Q) = F^0 \oplus G^0$, where $F^0$ and $G^0$ are the annihilators of $F$ and $G$, respectively. Let $\pi_{F^0} : T^*(T^*Q) \rightarrow F^0$ and $\pi_{G^0} : T^*(T^*Q) \rightarrow G^0$ be the projections with kernels $G^0$ and $F^0$, respectively.
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be the projections with kernels $G^0$ and $F^0$, respectively. We have, for $z \in T^*Q$, $\eta$ and $\zeta \in T^*(T^*Q)$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)).$$
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We have, for $z \in T^*Q$, $\eta$ and $\zeta \in T^*(T^*Q)$,

$$\Lambda_D(\eta, \zeta) = \Lambda(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)).$$

Therefore

$$\Lambda_D^\# = t \pi_{F^0} \circ \Lambda^\# \circ \pi_{F^0},$$

where the transpose $t \pi_{F^0} : G \to T(T^*Q)$ of $\pi_{F^0} : T^*(T^*Q) \to F^0$ is the canonical injection ($F^0$ being identified with the dual of $G$).
Dirac has proven that $\Lambda_D$ is a Poisson bivector. That property is a special case of the following proposition.
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**Proposition**  Let $(M, \omega)$ be a symplectic manifold, $F$ a symplectic vector subbundle of $TM$ and $G = \text{orth} F$ its symplectic orthogonal. Let

$$\Lambda^\#_D = t_{\pi_{F^0}} \circ \Lambda^\# \circ \pi_{F^0},$$

where $\Lambda$ is the Poisson bivector associated to $\omega$, $\pi_{F^0}$ defined as above. Then $\Lambda_D$ is a Poisson bivector field if and only if $G$ is completely integrable.
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**Proof** If $(M, \Lambda_D)$ is a Poisson manifold, $G$ is the vector subbundle tangent to its symplectic leaves. So it is completely integrable.
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**Proof** If $(M, \Lambda_D)$ is a Poisson manifold, $G$ is the vector subbundle tangent to its symplectic leaves. So it is completely integrable.

Conversely, assume that $G$ is completely integrable.
The Poisson-Dirac bivector (4)

We recall that if $\eta$ and $\zeta$ are two 1-forms on $M$, 

$$\Lambda_D(\eta, \zeta) = \Lambda(\pi_{F^0}(\eta), \pi_{F^0}(\zeta)) .$$

Let $\tau_M : TM \to M$ be the canonical projection of the tangent bundle. Then $(G, \tau_M|_G, M)$ is a Lie algebroid (with the bracket of vector fields, restricted to sections of $\tau_M|_G$ as composition law). Therefore the total space $G^*$ of its dual bundle has a linear Poisson structure. But we know that $G^*$ can be isentified with $F^0$, so $F^0$ has a linear Poisson structure. It means that the bracket (calculated for the Poisson structure $\Lambda$) of two 1-forms $\eta$ and $\zeta$ on $M$ which are sections of $F^0$ is again a section of $F^0$. The above formula for $\Lambda_D(\eta, \zeta)$ shows that $\Lambda_D$ is a Poisson bivector field. $\square$
Under the assumptions of the above Proposition, we have two Poisson structures $\Lambda$ and $\Lambda_D$ on $M$. Generally speaking, these two Poisson structures are not compatible.
Compatibility of $\Lambda$ and $\Lambda_D$

Under the assumptions of the above Proposition, we have two Poisson structures $\Lambda$ and $\Lambda_D$ on $M$. Generally speaking, these two Poisson structures are not compatible.

We have

$$\Lambda^\# - \Lambda_D^\# = t \pi_{G^0} \circ \Lambda^\# \circ \pi_{G^0},$$

and the same proposition shows that $\Lambda - \Lambda_D$ is a Poisson bivector if and only if $F$ is completely integrable.
Under the assumptions of the above Proposition, we have two Poisson structures \( \Lambda \) and \( \Lambda_D \) on \( M \). Generally speaking, these two Poisson structures are not compatible.

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\]

and the same proposition shows that \( \Lambda - \Lambda_D \) is a Poisson bivector if and only if \( F \) is completely integrable.

When both \( G \) and \( F \) are completely integrable the manifold \( M \) is locally a product of two symplectic manifolds.
Example (1)

On $T\mathbb{R}^n$, with coordinates $(q^i, \dot{q}^i)$, $1 \leq i \leq n$, we take as Lagrangian

$$L_0(q, \dot{q}) = \frac{m}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 - V(q),$$

and we impose the constraint

$$F(q) = \text{constant}.$$

We add one dimension to the configuration manifold (coordinate $\lambda$). So we have two more dimensions on $T(\mathbb{R}^n \times \mathbb{R})$, with coordinates $(\lambda, \dot{\lambda})$. Our new Lagrangian is

$$L(q, \lambda, \dot{q}, \dot{\lambda}) = L_0(q, \dot{q}) + \dot{\lambda}F(q) = \frac{m}{2} \sum_{i=1}^{n} (\dot{q}^i)^2 - V(q) + \dot{\lambda}F(q).$$
Example (2)

The Lagrange equations

\[
\begin{align*}
\frac{d}{dt}(mq^i) + \frac{\partial V(q)}{\partial q^i} - \lambda \frac{\partial F(q)}{\partial q^i} &= 0, \\
\frac{d}{dt} F(q) &= 0,
\end{align*}
\]

are the correct equations of motion of a heavy point constrained, by an ideal constraint, on a surface $F(q) = \text{constant}$, that constant depending on the initial condition. The Lagrange multiplier is $\lambda$. 
Example (2)

The Lagrange equations

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\begin{cases}
\frac{d}{dt}(m\dot{q}^i) + \frac{\partial V(q)}{\partial q^i} - \dot{\lambda} \frac{\partial F(q)}{\partial q^i} = 0, \\
\frac{d}{dt} F(q) = 0,
\end{cases}
\]

are the correct equations of motion of a heavy point constrained, by an ideal contraint, on a surface \( F(q) = \text{constant} \), that constant depending on the initial condition.

The Lagrange multiplier is \( \dot{\lambda} \).

The Legendre map is

\[
\mathcal{L} : (q, \lambda, \dot{q}, \dot{\lambda}) \mapsto (q, \lambda, p, p\lambda), \quad \text{with}
\]

\[
p_i = m\dot{q}^i, \quad p\lambda = F(q).
\]
The Hamiltonian $H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda)$, defined on $T\mathbb{R}^{n+1} \oplus T^*\mathbb{R}^{n+1}$, is

$$H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda) = \sum_{i=1}^{n} (p_i - \frac{m}{2} \dot{q}^i) \dot{q}^i + \dot{\lambda}(p_\lambda - F(q)) + V(q).$$
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As (non unique) Hamiltonian defined on $T^*\mathbb{R}^{n+1}$, we choose

$$\hat{H}(q, \lambda, p, p\lambda) = \frac{1}{2m} \sum_{i=1}^{n} p_i^2 + V(q).$$
Example (3)

The Hamiltonian $H(q, \lambda, \dot{q}, \dot{\lambda}, p, p_\lambda)$, defined on $T\mathbb{R}^{n+1} \oplus T^*\mathbb{R}^{n+1}$, is

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As (non unique) Hamiltonian defined on $T^*\mathbb{R}^{n+1}$, we choose

$$\hat{H}(q, \lambda, p, p_\lambda) = \frac{1}{2m} \sum_{i=1}^{n} p_i^2 + V(q).$$

We have only one primary constraint

$$\Phi(q, \lambda, p, p_\lambda) = F(q) - p_\lambda = 0,$$

with constraint function $\Phi = F(q) - p_\lambda$. 
The generalized Hamilton’s equation for the time derivative of any smooth function $g$ is

$$\frac{dg}{dt} = \{\hat{H}, g\} + v\{\Phi, g\}.$$
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The compatibility condition $\frac{d\Phi}{dt} = 0$ yields

$$\frac{1}{m} \chi(q, \lambda, p, p\lambda) = \frac{1}{m} \sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i} = 0.$$
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$$\frac{dg}{dt} = \{\hat{H}, g\} + v\{\Phi, g\}.$$ 

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$$\frac{1}{m}\chi(q, \lambda, p, p\lambda) = \frac{1}{m}\sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i} = 0.$$ 

We obtain a secondary constraint $\chi = 0$, with constraint function

$$\chi = \sum_{i=1}^{n} p_i \frac{\partial F(q)}{\partial q^i}.$$
So we get another compatibility condition \( \frac{d\chi}{dt} = 0 \), which leads to

\[
\sum_{i=1}^{n} \left( \frac{\partial F(q)}{\partial q^i} \right)^2 v = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \frac{\partial^2 F(q)}{\partial q^i \partial q^j} - \sum_{i=1}^{n} \frac{\partial F(q)}{\partial q^i} \frac{\partial V(q)}{\partial q^i}.
\]
Example (5)

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\]

This equality is not a new compatibility condition; it is a relation which determines \( v \) as a function of \( (q, \lambda, p, p_\lambda) \). We see that in fact \( v \) does not depend on \( \lambda \) nor on \( p_\lambda \).
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So we have a total of two constraint functions: \( \Phi \) and \( \chi \), with

\[
\{\Phi, \chi\} = -\sum_{i=1}^{n} \left( \frac{\partial F(q)}{\partial q^i} \right)^2.
\]

The constraints \( \Phi = 0 \) and \( \chi = 0 \) are second class.
Example (6)

The generalized Hamilton equations for the coordinates functions are

\[
\begin{align*}
\frac{dq^i}{dt} &= \frac{p_i}{m}, \\
\frac{dp_i}{dt} &= -\frac{\partial V(q)}{\partial q^i} - v(q, p) \frac{\partial F(q)}{\partial q^i}, \\
\frac{d\lambda}{dt} &= -v(q, p), \\
\frac{dp_\lambda}{dt} &= 0.
\end{align*}
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\end{align*}
\]

Remark

Instead of \( L(q, \lambda, \dot{q}, \dot{\lambda}) = L_0(q, \dot{q}) + \dot{\lambda} F(q) \), we may use as Lagrangian

\[
L(q, \lambda, \dot{q}, \dot{\lambda}) = L_0(q, \dot{q}) + \lambda (F(q) - C),
\]

where \( C \) is a constant. We obtain the same equations of motion on the constraint manifold \( F(q) = C \), and a similar generalized Hamiltonian formalism, with three constraints (one first class and two second class).
Example (7)

The Dirac brackets of the coordinates functions are

\[
\{ q^i, q^j \}_D = 0, \quad \{ q^i, \lambda \}_D = -\frac{1}{\{ \Phi, \chi \}} \frac{\partial F}{\partial q^i},
\]

\[
\{ q^i, p_j \}_D = -\delta^i_j - \frac{1}{\{ \Phi, \chi \}} \frac{\partial F}{\partial q^i} \frac{\partial F}{\partial q^j}, \quad \{ q^i, p_\lambda \}_D = 0,
\]

\[
\{ p_i, \lambda \}_D = \frac{1}{\{ \Phi, \chi \}} \sum_{k=1}^n p_k \frac{\partial^2 F}{\partial q^k \partial q^i},
\]

\[
\{ p_i, p_j \}_D = \frac{1}{\{ \Phi, \chi \}} \sum_{k=1}^n p_k \left( \frac{\partial F}{\partial q^j} \frac{\partial^2 F}{\partial q^k \partial q^i} - \frac{\partial F}{\partial q^i} \frac{\partial^2 F}{\partial q^k \partial q^j} \right),
\]

\[
\{ p_i, p_\lambda \}_D = 0, \quad \{ p_\lambda, \lambda \} = 1.
\]
Thanks

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References (1)


References (2)


