The works of Charles Ehresmann on connections: from Cartan connections to connections on fibre bundles, and some modern applications

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Élie Cartan’s affine connections (1)

Around 1923, Élie Cartan [1, 2, 3] introduced the notion of an *affine connection* on a manifold. That notion was previously used, in a less general setting, by H. Weyl [16] and rests on the idea of parallel transport due to T. Levi-Civita [11].

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Around 1923, Élie Cartan [1, 2, 3] introduced the notion of an *affine connection* on a manifold. That notion was previously used, in a less general setting, by H. Weyl [16] and rests on the idea of parallel transport due to T. Levi-Civita [11].

A large part of [1, 2] is devoted to applications of affine connections to Newtonian and Einsteinian Mechanics. Cartan show that the *principle of inertia* (which is at the foundations of Mechanics), according to which a material point particle, when no forces act on it, moves along a straight line with a constant velocity, can be expressed locally by the use of an affine connection. Under that form, that principle remains valid in (curved) Einsteinian space-times.

Élie Cartan's affine connections (2)

Cartan even shows that by a suitable adjustment of the connection, a gravity force (that means, an acceleration field) can be accounted for, and becomes a part of the Geometry of space-time. That result expresses the famous *equivalence principle* used by Einstein for the foundations of General Relativity. As shown by Cartan, it is valid for Newtonian Mechanics as well

Élie Cartan’s affine connections (3)

Cartan writes:

“Une variété à connexion affine est une variété qui, au voisinage immédiat de chaque point, a tous les caractères d’un espace affine, et pour laquelle on a une loi de repérage des domaines entourant deux points infiniment voisins: cela veut dire que si, en chaque point, on se donne un système de coordonnées cartésiennes ayant ce point pour origine, on connaît les formules de transformation (de même nature que dans l’espace affine) qui permettent de passer d’un système de référence à tout autre système de référence d’origine infiniment voisine”.

Élie Cartan's affine connections (4)

Approximate translation:
"A manifold with an affine connection is a manifold whose properties, in the neighborhood of each point, are those of an affine space, and on which there is a law for fitting together the neighborhoods of two infinitesimally nearby points: it means that if, on a neighborhood of each point, we have chosen Cartesian coordinates with that point as origin, we know the transformation formulae (of the same nature as those valid in an affine space) which allow to go from a reference frame to another reference frame with an infinitesimally nearby origin".

Élie Cartan's affine connections (5)

In fact, given a smooth manifold M, it is not on a neighborhood of each point $m \in M$ that Cartan considers a local affine strtucture. Rather, at each point $m \in M$, he considers the tangent space $T_m M$ endowed with its natural affine space structure. And he writes:

"La variété sera dite à *connexion affine* lorsqu'on aura défini, d'une manière d'ailleurs arbitraire, une loi permettant de repérer l'un par rapport à l'autre les espaces affines attachés à deux points infiniment voisins quelconques m et m' de la variété; cette loi permettra de dire que tel point de l'espace affine attaché au point m' correspond à tel point de l'espace affine attaché au point m, que tel vecteur du premier espace est parallèle ou équipollent à tel vecteur du second espace. En particulier le point m' lui-même sera repéré par rapport à l'espace affine du point $m \ldots$".

Élie Cartan's affine connections (6)

Approximate translation:

"The manifold will be said to be endowed with an *affine connection* once we have defined, in an arbitrary way, a law allowing to localize one with respect to the other the affine spaces attached to two infinitesimally nearby points m and m' of that manifold; that law will tell us which point of the affine space attached to m' corresponds to a given point of the space attached to m, and will tell us whether a vector living in the first space is parallel, or equipollent, to a vector living in the second space; in particular, the point m' itself will be localized in the affine space attached to $m \ldots$".

Élie Cartan's affine connections (7)

For a 3-dimensional manifold M, Cartan considers, at each point $m \in M$, an affine frame of the affine tangent space $T_m M$, with as origin the point m itself (identified with the null vector at m), and with the linear basis (e_1, e_2, e_3) as basis. In order to define the law which links the affine spaces tangent to the manifold M at two infinitesimally nearby points m and m', he write the relations

$$dm = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3 \,,$$

$$de_i = \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3 \,.$$

Élie Cartan's affine connections (8)

These equations mean that the point m', origin of $T_{m'}M$, infinitesimally near m, must be identified with the point

$$m + \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3$$

of the affine space T_mM.

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Similarly, the vectors e'_1, e'_2, e'_3 of $T_{m'}M$ must be identified with the vectors

$$e'_i = e_i + \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3$$

of T_mM.

Élie Cartan's affine connections (9)

The equations

$$dm = \omega^1 e_1 + \omega^2 e_2 + \omega^3 e_3 \,,$$
$$de_i = \omega_i^1 e_1 + \omega_i^2 e_2 + \omega_i^3 e_3 \,.$$

must be understood as equalities between differential 1-forms on a 15-dimensional space, with, as coordinates, the 3 coordinates which specify a point on M, and 12 more coordinates on which depend the affine frames of a 3-dimensional affine space. In fact, these differential 1-forms live on the principal bundle of affine frames of the affine tangent spaces to the maniflold M. This will become clear with the works of Charles Ehresmann [7].

Élie Cartan's affine connections (10)

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Other types of connections were defined and discussed by É. Cartan [3, 4, 5], with transformation groups which are no more subgroups of the group of affine transformations: conformal connections, projective connections, . . .

Conformal connections

É. Cartan [4] considers an n-dimensional manifold M and writes: "Attachons à chaque point P de cette variété un espace conforme à n dimensions, ... La variété sera dite *à connexion conforme* si nous nous donnons une loi (d'ailleurs arbitraire) permettant de rapporter, d'une manière conforme, l'espace conforme attaché au point P de la variété à l'espace conforme attaché au point infiniment voisin P' ".

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Approximate translation: “Let us link an n-dimensional conformal space to each point P of our manifold. That manifold will be said to be endowed with a *conformal connection* when we have specified, in an arbitrary way, how to tie (or maybe identify) the conformal space linked at point P with the conformal space linked to the infinitesimally nearby point P' ”.

Projective connections (1)

In [4], Cartan writes: “L’idée fondamentale se rattache à la notion de *parallélisme* que M. T. Levi-Civita a introduite de manière si féconde. Les nombreux auteurs qui ont généralisé la théorie des espaces métriques sont tous partis de l’idée fondamentale de M. Levi-Civita, mais, semble-t-il, sans pouvoir la détacher de l’idée de *vecteur*. Cela n’a aucun inconvénient quand il s’agit de variétés à connexion affine . . . Mais cela semblait interdire tout espoir de fonder une théorie *autonome* de variétés à connexion conforme ou projective. En fait, ce qu’il y a d’essentiel dans l’idée de M. Levi-Civita, c’est qu’elle donne un moyen pour raccorder entre eux deux petits morceaux infiniment voisins d’une variété, et c’est cette idée de *raccord* qui est féconde”.

Projective connections (2)

Approximate translation: “The fundamental idea stems from the notion of *parallelism* introduced by M. T. Levi-Civita in such a fruitful way. The many authors who genralized the theory of metric spaces all started from the fundamental idea of M. Levi-Civita, but, seemingly, without freeing it from the idea of *vector*. That does not matter as long as one deals with manifolds with affine connections . . . But that seemed to forbid any hope to build an *autonomous* theory of manifolds with conformal or projective connections. In fact, the main thing in M. Levi-Civita’s idea is that it allows to glue together two small, infinitesimally nearby pieces of a manifold, and it is that idea of gluing which is most fruitful”.

Projective connections (3)

The general idea underlying the notions of conformal connections, projective connections, and other types of connections on a manifold M is to use, as a local model of that manifold, an homogeneous space of the same dimension as M. A copy of that homogeneous space is attached at each point of the manifold, and considered as "tangent" at at that point to that manifold. The connection is essentially a law which indicates how these homogeneous spaces are glued together.

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That process was used by É. Cartan [1, 2, 3, 4]. Around 1950, it was fully formalized by C. Ehresmann [7]

Ehresmann connections (1)

Let $E(B, F)$ be a locally trivial differentiable fibre bundle with base B, standard fibre F, and canonical projection $\pi : E \to B$. For each $x \in B$, the fibre at x, $E_x = \pi^{-1}(x)$, is diffeomorphic to F. Ehresmann [7] defines an *infinitiesimal connection* on that bundle as a vector sub-bundle C of TE, complementary to $\ker(T\pi)$, *i.e* such that for each $z \in E$,

$$T_z E = \ker(T_z \pi) \oplus C_z \,,$$

which satisfies the additional condition:

given any smooth path $t \mapsto x(t)$ in B going from a point $x_0 = x(t_0)$ to another point $x_1 = x(t_1)$, and any $z_0 \in E_{x_0}$, there exists a smooth path $t \mapsto z(t)$ in E such that

$$z(t_0) = z_0 \quad \text{and} \quad \pi\bigl(z(t)\bigr) = x(t) \quad \text{for all} \quad t \,.$$

Ehresmann connections (2)

The map $z_0 = z(t_0) \mapsto z(t_1)$ is a diffeomorphism of the fibre E_{x_0} onto the fibre E_{x_1}. It is called the *parallel transport* of the fibres of $E(B, F)$ along the smooth path $t \mapsto x(t)$. Therefore the connection C determines a homomorphism of the groupoid of smooth paths in B into the groupoid of diffeomorphisms of a fibre or $E(B, F)$ onto another fibre.

Ehresmann connections (3)

Now let us assume that $E(B, F)$ is in fact a fibre bundle $E(B, F, G, H)$ with a Lie group G as structure group, H being the total space of the corresponding principal bundle $H(B, G, G_\gamma, \overline{H})$. An element h of H_x is a diffeomorphism from F onto E_x, and another diffeomorphism h' from F onto E_x is an element of H_x if and only if ${h'}^{-1} \circ h \in G$.

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The connection C is said to be *compatible with the structure group* G if the parallel transport along any smooth path in B starting from x_0 and ending at x_1, is of the form $h_1 \circ h_0^{-1}$, with $h_0 \in H_{x_0}$ and $h_1 \in H_{x_1}$.

Ehresmann connections (4)

Such a connection automatically determines a connection $\overline{C}$ on the principal bundle $H(B, G, G_\gamma, \overline{H})$, compatible with its structure group G_γ. Conversely, a connection $\overline{C}$ on the principal bundle $H(B, G, G_\gamma, \overline{H})$ compatible with its structure group determines a connection C on $E(B, F, G, H)$, compatible with its structure group.

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A connection $\overline{C}$ on the principal bundle $H(B, G, G_\gamma, \overline{H})$, compatible with its structure Lie group G, can be defined by a *connection form* ω. It is a 1-form on H, with values in the Lie algebra $\mathcal{G}$ of G, such that $\ker\omega = \overline{C}$ and that for each $X \in \mathcal{G}$, $h \in H$,

$$\big\langle \omega(h), X_H(h) \big\rangle = X \,,$$

where X_H is the fundamental vector field on H associated to X.

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Conversely, any 1-form ω on H which satisfies these two conditions is the 1-form of a connection $\overline{C}$ on H compatible with its structure Lie group.

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– there exists a smooth section $s_0 : B \to E$ of the bundle $E(B, F, G, H)$; we will identify B with the submanifold $s_0(B) \subset E$;

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Let H' be the subset of elements $h' \in H$ which map the point $o \in F$ into $s_0(B)$. Then $H'(B, G', G'_\gamma, \overline{H'})$ is a principal bundle with G' as structure Lie group. The connection C is called a *Cartan connection* if the form $\omega_{H'}$, induced on H' by the connection form ω, nowhere vanishes.

Cartan connections seen by Ehresmann

On a fibre bundle $E(B,F,G,H)$ which satisfies the conditions
– there exists a smooth section $s_0 : B \to E$,
– $F = G/G'$ is an homogeneous space of G,
– $\dim F = \dim B$,
there exists a Cartan connection if and only if the tangent bundle TB is isomorphic (as a vector bundle) to the bundle $T'B$ of vertical vectors, tangent to the fibres of E along $s_0(B)$. When such a vector bundle isomorphism exists, each fibre F_x of E will be said to be *tangent* at x to the base B, and the bundle $E(B,F,G,H)$ will be said to be *soldered* to B, by means of that isomorphism.

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A Cartan connection on $E(B, F, G, H)$ determines a vector bundle isomorphism $\chi_C : TB \to T'B$, therefore defines a soldering of that bundle to the base B.

Cartan connections seen by Ehresmann

That isomorphism χ_C is obtained as follows: take $x \in B$ and $v \in T_x B$. Choose any $h' \in H'_x$ and any $w \in T_{h'} H'$ which projects onto v. Then $\omega_{H'}(w)$ is in $\mathcal{G}$, identified with $T_e G$. Project it on $T_o F$ by the canonical map $G \to F = G/G'$, extended to vectors: we obtain a vector $\widetilde{w} \in T_o F$. Then h' is a diffeomorphism which maps the standard fibre F onto the fibre F_x of $E(B, F, G, H)$, and such that $h'(o) = s_o(x)$. Therefore $\chi_C(v) = Th'(\widetilde{w})$ is a vector tangent to the fibre F_x at $s_0(x)$, which depends only on x and v, not on h' or w. Then $\chi_C(v) = Th'(\widetilde{w})$.

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It admits a very simple interpretation in terms of the intrinsic derivative: $v \in T_x B$ can be considered as tangent at $s_0(x)$ to $s_0(B) \subset E$ E; $\chi_C(v)$ is its projection on $T_{s_0(x)} F_x$, with respect to the direct sum decomposition
$T_{s_0(x)} E = T_{s_0(x)} F_x \oplus C\big(s_0(x)\big)$.

Cartan connections seen by Ehresmann

Remark 1 The existence of a Cartan connection implies that H' is parallelizable, since the projection $TH' \to H'$ and the map $\omega_{H'} : TH' \to \mathcal{G}$ determine an isomorphism of TH' onto $H' \times \mathcal{G}$.

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Remark 2 Affine, projective and conformal connections considered by Cartan are Cartan connections in the sense of Ehresmann, who has shown that such connections exist on any smooth manifold.

Recent applications of connections

The first very important application of the notion of connection is probably to be found in the theory of General Relativity, in which the Levi-Civita connection associated to the pseudo-Riemannian structure on Space-Time plays a key role. According to the title of his paper [1], “Sur les espaces à connexion affine et la théorie de la relativité généralisée”, É. Cartan was probably, at least in part, motivated by possible physical applications when he investigated the properties of connections.

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Let us now indicate some more recent applications of connections in Mechanics and mathematical Physics: geometric quantization, mechanical systems, Maxwell’s equations in electromagnetism and Yang-Mills fields.

Geometric quantization

Let (M, Ω) be symplectic manifold. A *prequantization* of that symplectic manifold is a principal bundle P with base M and with the circle $S_1 = U(1)$ as structure group, endowed with a connection 1-form ω whose curvature is Ω.

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Since the Lie algebra of S_1 can be identified with $\mathbb{R}$, the connection form ω can be considered as a *contact form* on P. According to a theorem by B. Kostant [9] and J.-M. Souriau [15], there exists a prequantization of (M, Ω) if and only if the cohomology class of Ω is integer.

Phases in Mechanics

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Let us consider a Hamiltonian system, depending of some parameters, which for any fixed value of these parameters, is completely integrable. The motion of the system, for a fixed value of the parameter, is quasi-periodic on a Lagrangian torus. of phase space. At a certain time, the parameters vary slowly, describe a closed loop in the space of values of the parameters, and after taking again their initial values, remain constant. The motion of the system becomes again quasi-periodic on the same Lagrangian torus, but with a change of phase (the Hannay and Berry phase). This change of phase is interpreted as the holonomy of an Ehresmann connection in the works of Marsden, Montgomery and Ratiu [13, 14].

Nonholonomic constraints(3)

Several different approaches have been used for the mathematical description of mechanical systems with constraints. In one of these approaches, the configuration space of the system is a smooth manifold and the constraints are described by a vector (or sometimes an affine) sub-bundle C of the tangent bundle TQ. The admissible motions of the system are smooth curves $t \mapsto x(t)$ in Q such that, for all t,

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J. Koiller [10] considered systems where the configuration space Q is the total space of a principal bundle over a base B, with a Lie group G as structure group, and where the constraint C is a connection on that principal bundle.

Active constraints (1)

Let us consider a mechanical system in which some geometric constraints can be acted on, as a function of time, in order to control the motion of the system. For example, a cat in free fall can change the shape of her body to try (generally with success) to reach the ground on her feet.

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For the mathematical description of such systems [12], we use a manifold Q as configuration space, and a smooth submersion $\pi : Q \to S$ onto another manifold S (the space of shapes of the cat's body, or more generally the space of possible states of the active constraint).

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For the mathematical description of such systems [12], we use a manifold Q as configuration space, and a smooth submersion $\pi : Q \to S$ onto another manifold S (the space of shapes of the cat's body, or more generally the space of possible states of the active constraint).

The dynamical properties of the system (other than those used to change the value of the active constraint) are described by a Lagrangian $L : TQ \to \mathbb{R}$. Let V^*Q be the dual bundle of the vertical sub-bundle $VQ = \ker T\pi \subset TQ$.

Active constraints (2)

V^*Q is the quotient bundle $T^*Q/(VQ)^0$. Let $\zeta : T^*Q \to V^*Q$, $q : V^*Q \to Q$ and $\widetilde{\pi} = \pi \circ q : V^*Q \to S$ be the projections. When the Lagrangian L is

$$L(v) = \frac{1}{2} g(v, v) - P(x)\,, \quad \text{with} \quad x \in Q\,,\ v \in T_x Q\,,$$

there is on the bundle $V^*Q \to S$ an Ehresmann connection (called the dynamical connection) which can be used to determine the way in which an infinitesimal change of the state of the connection, represented by a vector tangent to S, affects the motion of the mechanical system.

Active constraints (3)

Let us call *kinetic connection* the Ehresmann connection, on the bundle $\pi : Q \to S$, for which the horizontal lift at $x \in Q$ of a vector $v \in T_{\pi(x)}S$ is the unique $w \in T_xQ$, orthogonal (with respect to g) to the vertical subspace $\ker T_x\pi$, such that $T\pi(w) = v$. The dynamic connection is characterized by the two properties:

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(1) The horizontal lift at $z \in \widetilde{\pi}^{-1}(s)$ of a vector $v \in T_s S$ with respect to the dynamical connection projects on Q onto the horizontal lift at $x = q(z)$ of v with respect to the kinetic connection.

Active constraints (3)

Let us call *kinetic connection* the Ehresmann connection, on the bundle $\pi : Q \to S$, for which the horizontal lift at $x \in Q$ of a vector $v \in T_{\pi(x)}S$ is the unique $w \in T_x Q$, orthogonal (with respect to g) to the vertical subspace $\ker T_x\pi$, such that $T\pi(w) = v$. The dynamic connection is characterized by the two properties:

(1) The horizontal lift at $z \in \widetilde{\pi}^{-1}(s)$ of a vector $v \in T_s S$ with respect to the dynamical connection projects on Q onto the horizontal lift at $x = q(z)$ of v with respect to the kinetic connection.

(2) The horizontal lift to V^*Q of any smooth vector field on S, with respect to the dynamical connection, is an infinitesimal automorphism of the Poisson structure of V^*Q.

Maxwell's equations (1)

The famous Maxwell's equations are usually written (see for example [8])

$$\operatorname{rot}\overrightarrow{E}+\frac{\partial\overrightarrow{B}}{\partial t}=0\,,\quad \operatorname{rot}\overrightarrow{H}-\frac{\partial\overrightarrow{D}}{\partial t}=4\pi\overrightarrow{j}\,,$$
$$\operatorname{div}\overrightarrow{B}=0\,,\qquad\qquad \operatorname{div}\overrightarrow{D}=4\pi\rho\,,$$

where $\overrightarrow{E}$ is the electric field, $\overrightarrow{B}$ the magnetic induction, $\overrightarrow{D}$ the diplacement current, $\overrightarrow{H}$ the magnetic field, ρ the electric charge density and $\overrightarrow{j}$ the current density. Moreover there are constitutive equations which link $\overrightarrow{E}$ and $\overrightarrow{D}$, $\overrightarrow{B}$ and $\overrightarrow{H}$,

$$\overrightarrow{D}=\varepsilon_0\overrightarrow{E}\,,\quad \overrightarrow{H}=\frac{1}{\mu_0}\overrightarrow{B}\,.$$

Maxwell's equations (2)

Let us introduce the 2-form on space-time

$$F = B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 + (E_1 dx^1 + E_2 dx^2 + E_3 dx^3) \wedge dt\,,$$

and

$$G = D_1 dx^2 \wedge dx^3 + D_2 dx^3 \wedge dx^1 + D_3 dx^1 \wedge dx^2 - (H_1 dx^1 + H_2 dx^2 + H_3 dx^3) \wedge dt\,.$$

Let us set

$$J = \rho dx^1 \wedge dx^2 \wedge dx^3 - (j_1 dx^1 + j_2 dx^2 + j_3 dx^3) \wedge dt\,.$$

Maxwell's equations (3)

Then

$$G = \sqrt{\frac{\varepsilon_0}{\mu_0}} * F\,,$$

where $*$ is the Hodge operator on the 4-dimensional pseudo-Riemannian manifold Space-Time.

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Maxwell's equations become

$$dF = 0 \,, \quad dG = 4\pi J \,, \quad \text{with} \quad G = \sqrt{\frac{\varepsilon_0}{\mu_0}} * F \,.$$

Maxwell's equations (4)

Maxwell's equations can be made even more beautiful if the electromagnetic 2-form F on Space-Time can be considered as the curvature form of a connection on a principal bundle, with Space-Time as base and the circle $S_1 = U(1)$ as structure group (this idea was first introduced by H. Weyl). The 2-form F should then be considered as taking its values in the Lie algebra $u(1)$ of S_1. The connection form A such that $F = DA$, is not unique: we may add a closed 1-form (gauge transformation). The first Maxwell's equation, $dF = 0$, is automatically satisfied. The second Maxwell's equation becomes

$$D(*DA) = 4\pi\sqrt{\frac{\mu_0}{\varepsilon_0}}J\,,$$

where D is the covariant exterior differential operator.

Yang-Mills fields

Gauge theories generalize Maxwell's theory of electromagnetism written in terms of connections. They use a principal bundle with Space-Time as base and a non-Abelian group as structure group ($U(1) \times SU(2) \times SU(3)$ in the so-called *standard model*).

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They introduce a connection 1-form A on that bundle and lead, for the curvature form F of the connection, to field equations similar to Maxwell's equations:

$$D_A A = F\,,\quad D_A F = 0\quad \text{(Bianchi identity)}\,,$$

and

$$D_A * F = J\,,$$

where D_A is the covariant differential with respect to the connection A and J a "current" which generalizes the 4-dimensional current density of Maxwell's theory.

References (1)

References

[1] É. Cartan, Sur les espaces à connexion affine et la théorie de la relativité généralisée, partie I, *Ann. Ec. Norm., t. 40, 1923, p. 325–412.*

[2] É. Cartan, Sur les espaces à connexion affine et la théorie de la relativité généralisée, partie I (suite), *Ann. Ec. Norm., t. 41, 1921, p. 1–25.*

[3] É. Cartan, Sur les espaces à connexion affine et la théorie de la relativité généralisée partie II, *Ann. Ec. Norm.*, t. 42, 1925, p. 17–88.

[4] É. Cartan, Les espaces à connexion conforme, *Ann. Soc. Pol. Math.*, t. 2, 1923, p. 171–221.

References (2)

References

[5] É. Cartan, Sur les variétés à connexion projective, *Bull. Soc. Math. France*, t. 52, 1924, p. 205–241.

[6] É. Cartan, Les récentes généralisations de la notion d'espace, *Bull. Sc. Math.*, t. 48, 1924, p. 294–320.

[7] C. Ehresmann, Les connexions infinitésimales dans un espace fibré différentiable, *Colloque de topologie de Bruxelles*, 1950, p. 29–55.

[8] V. Guillemin and S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

References (3)

References

[9] B. Kostant, Quantization and unitary representations, in *Modern analysis and applications*,Lecture Notes in Math. vol. 170, Springer-Verlag, 1970, p. 101–184.

[10] J. Koiller, Reduction of some classical non-holonomic systems with symmetry*Arch. Ration. Mech. Anal.*, vol. 118, 1992, p. 113–148.

[11] T. Levi-Civita, *Rendiconti del Circ. matem. di Palermo*, t. 42, 1917, p. 173–205.

[12] C.-M. Marle, Géométrie des systèmes mécaniques à liaisons actives, in *Symplectic Geometry and Mathematical Physics* (P. Donato, J. Elhadad, G.M. Tuynman, editors), Birkhäuser, Boston, 1991, p. 260–287.

References (4)

References

[13] J. Marsden, R. Montgomery and T. Ratiu, *Reduction, Symmetry and Phases in Mechanics*, Memoirs of the American Mathematical Society, 88 (1988), p. 269–294.

[14] R. Montgomery, The connectiion whose holonomy is the classical adiabatic angles of Hannay and Berry and its generalization to the non-integrable case, *Commun. Math. Phys.* 120 (1988), 269–294.

[15] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1970.

[16] H. Weyl, *Temps, Espace, Matière* (traduction française), Librairie scientifique Albert Blanchard, Paris, 1958.