A direct proof of Malus' theorem using the symplectic sructure of the set of oriented straight lines

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Summary

What is Malus' theorem ?

In (non-relativistic) *Physics*, once a unit of length is chosen, the *Physical Space* in which we live is a *three-dimensional Euclidean affine space* $\mathcal{E}$.

In *Geometrical Optics*, the propagation of light is described in terms of *light rays* (in short *rays*). In an homogeneous medium, rays are *oriented straigh lines*.

The set of *all possible light rays*, identified with the set of all possible oriented straight lines, depends on *four parameters* : indeed an oriented straight line D is determined by

- the unit vector $\overrightarrow{u}$ parallel to and with the same orientation as D (2 parameters),
- and two more parameters, for example the coordinates of the intersection point of D with a plane not parallel to to $\overrightarrow{u}$, to fully determine the position of the straight line D.

Similarly, in an n-dimensional Euclidean affine space, the set of *all possible oriented straight lines* depends on $2(n-1)$ parameters.

What is Malus' theorem ? (2)

Definition

The *rank* of a family of light rays is the number of parameters of which this family depends.

Examples

The set of rays emitted by a *luminous point source* is a *rank 2* family of rays.

The set of *oriented normals* to a smooth surface too is a *rank 2* family of rays.

Definition

A rank 2 family of rays is said to be *normal* if through each point of a ray of the family, there exists a smooth surface such that *all the neighbouring rays which cross that surface cross it orthogonally*.

Example

Light rays emitted by a *luminous point source* : all the *spheres* centered on the point source are crossed by the rays orthogonally.

What is Malus' theorem ? (3)

Theorem (Malus' Theorem)

A two parameter normal family of light rays remains normal after any number of reflections on smooth reflecting surfaces or refractions across smooth surfaces which separate transparent media with different refractive indexes. No assumption is made about the shapes of these surfaces. It is only assumed that these surfaces are smooth and that the reflections or refractions obey the well-known laws of Optics.

History of Malus' theorem (1)

Étienne Louis Malus (1775–1812) (X 1794) is a French scientist who investigated geometric properties of families of straight oriented lines, in view of applications to light rays. He developed *Huygens* undulatory theory of light. He discovered and investigated the phenomenon of *polarization of light* and the phenomenon of *double refraction* of light in crystals. He participated in Napoleon's disastrous expedition into Egypt (1798 to 1801) where he contracted diseases responsible for his early death..

He proved that the family of rays emitted by a luminous point source (which, as we have seen above, is normal) remains normal after *one reflection* on a smooth surface, or *one refraction* through a smooth surface, but he was not sure whether these propertied remain true for *several* reflections or refractions. His works of families or oriented straight lines were later used and enhanced by *William Rowan Hamilton*

History of Malus' theorem (2)

A *full proof* of Malus' Theorem was obtained independently by the famous Irish mathematician *William Rowan Hamilton* (1805–1865) and by the French scientist *Charles Dupin* (1784–1873). In French Optics manuals, Malus' Theorem is frequently called Malus-Dupin's Theorem. I do not know whether Hamilton obtained his proof *before* or *after* Dupin. I have not read Dupin's proof, but I have read Hamilton's proof given in his paper *Theory of systems of rays* (1827). That proof rests on the *stationarity properties of the optical length* of rays, with respect to infinitesimal displacements of the points of reflections or of refractions, on the reflecting or refracting surfaces.

Charles François Dupin (1784–1873) (X 1801) is a French mathematician and naval engineer. It is said in *Wikipedia* that he inspired to the famous poet and novelist *Edgar Allan Poe* (1809–1849) the figure of *Auguste Dupin* who appears in his three detective stories : *The murders in the rue Morgue*, *The Mystery of Marie Roget*, *The Purloined Letter*.

A direct proof of Malus' theorem

Malus' Theorem can be proven by using the stationarity of the optical length measured between two points on the same light ray, with respect to infinitesimal displacements of the points of reflection or of refraction between these two points. Using that property, it can be proven that *starting from a surface orthogonal to the rays* and displacing it along the rays, the optical length by which it is displaced being the same for all rays, we get *another surface orthogonal to the rays*. This is true *even when during the displacement, refleting or refracting surfaces are encountered*. *Hamilton*'s proof rests on this idea.

I will propose now anoter proof of Malus' Theorem, which *directly uses the laws of reflection and refraction in Optics*. It is made in four steps.

A direct proof of Malus' theorem (2)

- I will first prove that the *set of all possible light rays* in an Euclidean 3-dimensional affine space has a *smooth symplectic manifold structure*, of dimension 4 (or, more generally, of dimension $2(n-1)$ if the Euclidean affine space is n-dimensional).
- Then, using the well known laws of Optics, I will prove that reflections and refractions are *symplectic diffeomorphisms*.
- Next, I will prove that a *rank 2 family of rays is normal* if and only if *it is a Lagrangian submanifold* of the symplectic manifold of all possible rays. (If the Euclidean space is n-dimensional, replace rank 2 by rank $n-1$).
- Finally, *Malus' Theorem easily follows*.

The symplectic structure of the set of all possible rays

Proposition

The set $\mathcal{L}$ of all possible oriented straight lines in the affine Euclidean space $\mathcal{E}$ is diffeomorphic to the cotangent bundle $T^*\Sigma$ to a sphere, by a symplectic diffeomorphism.

Proof

Let indeed Σ be a sphere of any fixed radius R (for example $R = 1$), centered on a point C, and O be another fixed point in $\mathcal{E}$. Of course we can take $O = C$, but for clarity it is better to separate these two points. An oriented straight line L determines

- an unique point $m \in \Sigma$ such that the vector $\overrightarrow{u} = \overrightarrow{Cm}$ is parallel to and of same direction as L,
- an unique linear form η on the tangent space $T_m\Sigma$ at m to the sphere Σ, given by

$$\eta(\overrightarrow{w}) = \overrightarrow{OP}.\overrightarrow{w} \quad \text{for all} \quad \overrightarrow{w} \in T_m\Sigma\,,$$

where P is any point of the line L.

The symplectic structure of the set of all possible rays (2)

The pair (m, η) is an element of the *cotangent bundle* $T^*\Sigma$. In fact m being determined by η, we can say that η is an element of $T^*\Sigma$.

Conversely, an element $\eta \in T^*\Sigma$, *i.e.* a linear form η on the tangent space to Σ at some point $m \in \Sigma$, determines an oriented straight line L, parallel to and of the same direction as $\overrightarrow{u} = \overrightarrow{Cm}$. This line is the set of points $P \in \mathcal{E}$ such that

$$\overrightarrow{OP}.\overrightarrow{w} = \eta(\overrightarrow{w}) \quad \text{for all } \overrightarrow{w} \in T_m\Sigma\,.$$

There exists on the cotangent bundle $T^*\Sigma$ a unique *differential one-form* λ_Σ, called the *Liouville form*, whose exterior differential $\mathrm{d}\lambda_\Sigma$ is a *symplectic form* on $T^*\Sigma$. The above described diffeomorphism between the set $\mathcal{L}$ of all oriented straight lines and the cotangent bundle $T^*\Sigma$ allows us to transport on $\mathcal{L}$ the Liouville one form λ_Σ and the symplectic form $\mathrm{d}\lambda_\Sigma$. So we get on $\mathcal{L}$ a differential one-form λ_O and a symplectic form $\omega = \mathrm{d}\lambda_O$. Therefore $(\mathcal{L}, \omega)$ is a *symplectic manifold*.

The symplectic structure of the set of all possible rays (3)

The diffeomorphism so obtained, the one-form λ_O and its exterior differential $\omega = \mathrm{d}\lambda_O$ *do not depend on* the choice of the centre C of the sphere Σ (with the obvious convention that two spheres of the same radius centered on two different points C and C' are identified by means of the translation which sends C on C').

However, this diffeomorphism *depends on* the choice of the point O, and so does the one-form λ_O : when, to a given straight line L, the choice of O associates the pair $(m, \eta) \in T^*\Sigma$, the choice of *another point* O' associates the pair $\bigl(m, \eta + \mathrm{d}f_{O'O}(m)\bigr)$, where $f_{O'O} : \Sigma \to \mathbb{R}$ is the smooth function defined on Σ

$$f_{O'O}(n) = \overrightarrow{O'O}.\overrightarrow{Cn}\,, \quad n \in \Sigma\,.$$

Therefore, if the choice of O determines on $\mathcal{L}$ the one-form λ_O, the choice of O' determines the one-form

$$\lambda_{O'} = \lambda_O + \mathrm{d}(f_{O'O} \circ \pi_\Sigma)$$

where $\pi_\Sigma : T^*\Sigma \to \Sigma$ is the canonical projection.

The symplectic structure of the set of all possible rays (4)

The symplectic form ω on the set of all oriented straight lines $\mathcal{L}$ *does not depend on the choice of O, nor on the choice of C*, since we have

$$\omega = \mathrm{d}\lambda_O = \mathrm{d}\lambda_{O'} \quad \text{because } \mathrm{d}\circ\mathrm{d} = 0\,.$$

Proposition

Let $(\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ be an orthonormal basis of the Euclidean vector space $\overrightarrow{\mathcal{E}}$ associated to the affine Euclidean space $\mathcal{E}$. Any oriented straight line $L \in \mathcal{L}$ can be determined by its unit directing vector $\overrightarrow{u}$ ans by a point $P \in L$ (determined up to addition of a vector collinear with $\overrightarrow{u}$). Expressed in terms of the coordinates (p_1, p_2, p_3) of P in the affine frame $(O, \overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3)$ and of the components (u_1, u_2, u_3) of $\overrightarrow{u}$, the symplectic form ω is given by

$$\omega = \sum_{i=1}^{3} \mathrm{d}p_i \wedge \mathrm{d}u_i\,.$$

Proof. Using the definition of the Liouville one-form on $T^*\Sigma$, we see that

$$\lambda_0 = \sum_{i=1}^{3} p_i \mathrm{d}u_i \,.$$

Therefore

$$\omega = \mathrm{d}\lambda_O = \sum_{i=1}^{3} \mathrm{d}p_i \wedge \mathrm{d}u_i \,.$$

Remark

The three components u_1, u_2, u_3 of $\overrightarrow{u}$ are not independent, since they must satisfy $\sum_{i=1}^{3}(u_i)^2 = 1$. The point $P \in L$ used to detemine the oriented straight line L is not uniquely determined, since by adding to P any vector collinear with $\overrightarrow{u}$ we get another point in L. However, these facts do not affect the validity of the expression of ω given above.

The symplectic structure of the set of all possible rays (6)

Remark

The symplectic form ω can be expressed very concisely by using an obvious vector notation combining the wedge and scalar products :

$$\omega(P, \overrightarrow{u}) = \mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\overrightarrow{u}\,.$$

Reflection is a symplectic diffeomorphism

Proposition

Let M be a smooth reflecting surface. Let Reflx_M be the map which associates to each light ray L_1 which hits M on its reflecting side, the reflected light ray $L_2 = \mathrm{Reflx}_M(L_1)$. The map Reflx_M is a symplectic diffeomorphism defined on the open subset of the symplectic manifold $(\mathcal{L}, \omega)$ made by light rays which hit Mon its reflecting side, onto the open subset made by the same straight lines with the opposite orientation.

Proof. Any oriented straight line L_1 which hits the mirror M is determined by

- the unit vector $\overrightarrow{u}_1$ parallel to and of same direction as L_1,
- the incidence point $P \in M$ of the light ray on the mirror.

We will write $\overrightarrow{P}$ for the vector $\overrightarrow{OP}$, the fixed point O being arbitrarily chosen.

Reflection is a symplectic diffeomorphism (2)

The reflected ray L_2 is determined by

- the unit vector $\overrightarrow{u}_2$, given in terms of $\overrightarrow{u}_1$ by the formula
$$\overrightarrow{u}_2 = \overrightarrow{u}_1 + 2(\overrightarrow{u}_1.\overrightarrow{n})\overrightarrow{n}\,,$$
where $\overrightarrow{n}$ is a unit vector normal to the mirror M at the incidence point P, with anyone of the two possible orientations ;
- the same point $P \in M$ on the mirror.

According to the expression of the symplectic form ω given in the last Remark, we have to check that $\mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\overrightarrow{u}_2 = \mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\overrightarrow{u}_1$. We have

$$\begin{aligned}\mathrm{d}\overrightarrow{P} \wedge \mathrm{d}(\overrightarrow{u}_2 - \overrightarrow{u}_1) &= 2\mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\bigl((\overrightarrow{u}_1.\overrightarrow{n})\overrightarrow{n}\bigr)\\ &= -2\mathrm{d}\bigl((\overrightarrow{u}_1.\overrightarrow{n})(\overrightarrow{n}.\mathrm{d}\overrightarrow{P})\bigr)\\ &= 0\,,\end{aligned}$$

because $\overrightarrow{n}.\mathrm{d}\overrightarrow{P} = 0$, the differential $\mathrm{d}\overrightarrow{P}$ lying tangent to the mirror M, while the vector $\overrightarrow{n}$ is normal to the mirror.

Refraction is a symplectic diffeomorphism

Proposition

Let R be a smooth refracting surface, which separates two transparent media with refractive indexes n_1 and n_2. Let Refrac_R be the map which associates, to each light ray L_1 which hits the refracting surface R on the side of refracting index n_1, the corresponding refracted ray $L_2 = \mathrm{Refrac}_R(L_1)$ determined by Snell's law of refraction. The map Refrac_R is a symplectic diffeomorphism defined on an open subset of $(\mathcal{L}, n_1\omega)$, (the set of oriented straight lines which hit R on the n_1 side and, if $n_1 > n_2$, are not totally reflected) with values in an open subset of $(\mathcal{L}, n_2\omega)$.

Proof. Any oriented straight line L_1 which hits the refracting surface R is determined by

- the unit vector $\overrightarrow{u}_1$ parallel to and of same direction as L_1,
- the incidence point $P \in R$ of the light ray on the refracting surface.

We will write $\overrightarrow{P}$ for the vector $\overrightarrow{OP}$, the fixed point O being arbitrarily chosen.

Refraction is a symplectic diffeomorphism (2)

The refracted ray L_2 is determined by

- the unit vector $\overrightarrow{u}_2$, related to $\overrightarrow{u}_1$ by the formula

$$n_2\left(\overrightarrow{u}_2 - (\overrightarrow{u}_2.\overrightarrow{n})\overrightarrow{n}\right) = n_1\left(\overrightarrow{u}_1 - (\overrightarrow{u}_1.\overrightarrow{n})\overrightarrow{n}\right),$$

  where $\overrightarrow{n}$ is a unit vector normal to the refractig surface R at the incidence point P, with anyone of the two possible orientations ;
- the same point $P \in R$ on the refracting surface.

We have to check that $n_2 \mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\overrightarrow{u}_2 = n_1 \mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\overrightarrow{u}_1$. We have

$$\begin{aligned}
\mathrm{d}\overrightarrow{P} \wedge (n_2 \mathrm{d}\overrightarrow{u}_2 - n_1 \mathrm{d}\overrightarrow{u}_1) &= \mathrm{d}\overrightarrow{P} \wedge \mathrm{d}\left(n_2(\overrightarrow{u}_2.\overrightarrow{n})\overrightarrow{n} - n_1(\overrightarrow{u}_1.\overrightarrow{n})\overrightarrow{n}\right)\\
&= -\mathrm{d}\Bigl(\bigl(n_2(\overrightarrow{u}_2.\overrightarrow{n}) - n_1(\overrightarrow{u}_1.\overrightarrow{n})\bigr)(\overrightarrow{n}.\mathrm{d}\overrightarrow{P})\Bigr)\\
&= 0,
\end{aligned}$$

because $\overrightarrow{n}.\mathrm{d}\overrightarrow{P} = 0$, the differential $\mathrm{d}\overrightarrow{P}$ lying tangent to the refracting surface R, while the vector $\overrightarrow{n}$ is normal to R.

Normal systems are Lagrangian submanifolds

Proposition

A *rank 2 family* *of oriented straight lines, i.e. a family depending smoothly on* 2 *parameters, is* *normal* *if and only if it is a* *Lagrangian submanifold* *of the symplectic manifold* $(\mathcal{L}, \omega)$ *of all oriented straight lines.*

Proof. Let us consider a rank 2 family of oriented straight lines. Locally, in a neighbourhood of each of its straight lines, the family can be determined by a smooth map $(k_1, k_2) \mapsto L(k_1, k_2)$, defined on an open substet of $\mathbb{R}^2$, with values in the manifold $\mathcal{L}$ of oriented straight lines. For each value of (k_1, k_2), the ray $L(k_1, k_2)$ can be determined by

- a point $P(k_1, k_2)$ of the ray $L(k_1, k_2)$,
- the unit director vector $\overrightarrow{u}(k_1, k_2)$ of the ray $L(k_1, k_2)$

Although $P(k_1, k_2)$ is not uniquely determined, we can arrange things so that the map $(k_1, k_2) \mapsto \big(P(k_1, k_2), \overrightarrow{u}(k_1, k_2)\big)$ is smooth. By assumption it is everywhere of rank 2.

Normal systems are Lagrangian submanifolds (2)

The reciprocal image of the symplectic form ω of $\mathcal{L}$ by the map $(k_1, k_2) \mapsto \big(P(k_1, k_2), \overrightarrow{u}(k_1, k_2)\big)$ is

$$\left(\frac{\partial \overrightarrow{P}(k_1, k_2)}{\partial k_1} \frac{\partial \overrightarrow{u}(k_1, k_2)}{\partial k_2} - \frac{\partial \overrightarrow{P}(k_1, k_2)}{\partial k_2} \frac{\partial \overrightarrow{u}(k_1, k_2)}{\partial k_1}\right) \mathrm{d}k_1 \wedge \mathrm{d}k_2 \,,$$

where, as before, we have written $\overrightarrow{P}(k_1, k_2)$ for $\overrightarrow{OP}(k_1, k_2)$, the origin O being any fixed point in $\mathcal{E}$. Using the symmetry property of the second derivatives

$$\frac{\partial^2 \overrightarrow{P}(k_1, k_2)}{\partial k_1 \partial k_2} = \frac{\partial^2 \overrightarrow{P}(k_1, k_2)}{\partial k_2 \partial k_1}$$

we see that the reciprocal image of ω can be written

$$\left(\frac{\partial}{\partial k_2}\left(\overrightarrow{u} . \frac{\partial \overrightarrow{P}}{\partial k_1}\right) - \frac{\partial}{\partial k_1}\left(\overrightarrow{u} . \frac{\partial \overrightarrow{P}}{\partial k_2}\right)\right) \mathrm{d}k_1 \wedge \mathrm{d}k_2 \,.$$

where we have written $\overrightarrow{u}$ and $\overrightarrow{P}$ for $\overrightarrow{u}(k_1, k_2)$ and $\overrightarrow{P}(k_1, k_2)$.

Normal systems are Lagrangian submanifolds (3)

Our rank 2 family of rays is a Lagrangian submanifold of $(\mathcal{L}, \omega)$ is and only if the reciprocal image of ω vanishes, *i.e.*, if and only if

$$\frac{\partial}{\partial k_2}\left(\overrightarrow{u}.\frac{\partial \overrightarrow{P}}{\partial k_1}\right) = \frac{\partial}{\partial k_1}\left(\overrightarrow{u}.\frac{\partial \overrightarrow{P}}{\partial k_2}\right),$$

or if and only if there exists locally a smooth function $(k_1, k_2) \mapsto F(k_1, k_2)$ such that

$$\overrightarrow{u}.\frac{\partial \overrightarrow{P}}{\partial k_1} = \frac{\partial F}{\partial k_1}, \quad \overrightarrow{u}.\frac{\partial \overrightarrow{P}}{\partial k_2} = \frac{\partial F}{\partial k_2}. \qquad (*)$$

Let us now look at the necessary and sufficient conditions under which there exists locally, near a given ray of the family, there exists a smooth surface orthogonal to all the neighbouring rays of the family. This surface is the image of a map

$$(k_1, k_2) \mapsto P(k_1, k_2) + \lambda(k_1, k_2)\overrightarrow{u}(k_1, k_2),$$

where $(k_1, k_2) \mapsto \lambda(k_1, k_2)$ is a smooth function.

Normal systems are Lagrangian submanifolds (4)

This surface is orhtogonal to the rays if and only if the function λ is such that

$$\overrightarrow{u}(k_1, k_2).\mathrm{d}\bigl(\overrightarrow{P}(k_1, k_2) + \lambda(k_1, k_2)\overrightarrow{u}(k_1, k_2)\bigr) = 0\,.$$

By using the equalities $\overrightarrow{u}(k_1, k_2).\mathrm{d}\overrightarrow{u}(k_1, k_2) = 0$ and $\overrightarrow{u}(k_1, k_2).\overrightarrow{u}(k_1, k_2) = 1$, this condition becomes

$$\overrightarrow{u}(k_1, k_2).\mathrm{d}\overrightarrow{P}(k_1, k_2) + \mathrm{d}\lambda(k_1, k_2) = 0\,. \qquad (**)$$

We see that when there exists a smooth function F which satifies $(*)$, all functions $\lambda = -F+$ Constant satisfy $(**)$, and conversely when there exists a smooth function λ which satisfies $(**)$, all functions $F = -\lambda +$ Constant satisfy $(*)$. A rank 2 family of rays is therefore normal il and only if it is a Lagrangian submanifold of $\mathcal{L}$.

Proof of Malus' Theorem

Since *reflections* and *refractions* are *symplectic diffeomorphisms*, and since by composing several symplectic diffeomorphisms we get again a *symplectic diffeomorphism*, the travel of light rays through an optical device with any number of reflecting or refracting smooth surfaces is a *symplectic diffeomorphism*.

The image of a *Lagrangian submanifold* by a symplectic diffeomorphism is automatically a *Lagrangian submanifold*.

This proves *Malus' Theorem*.

Thanks

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